Network Basics 1

Graph ML

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Lecture Topics

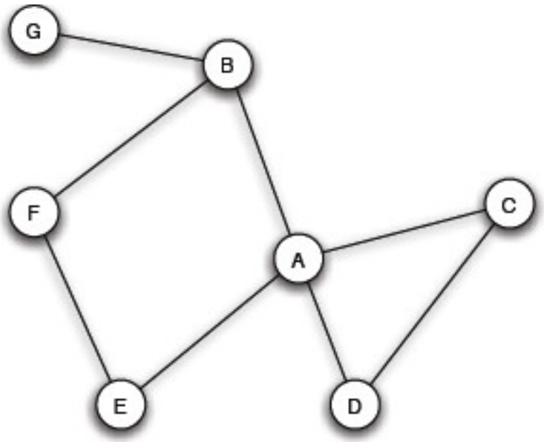
- Graph Theory
    - Node degree
    - Graph density
    - Complete Graph
    - Distance and Diameter
    - Adjacency matrix
    - Graph Connectivity
    - Reachability
    - Sub-graphs
    - Graph Types

Graph Theory

- A graph consists of
  - **N**: a set of nodes (items, entities, people, etc), and
  - **E**: a set of links or edges between nodes

- Graph is a way to **specify relationships** / links amongst a set of nodes.

- We define
  - N=|**N**| → size of **N**
  - E=|**E**| → size of **E**



Graph Theory. Cnt.

- Nodes *i* and *j* are *adjacent* or *neighbors* if:
  - There is an edge btw them!
    - $i \in \mathbf{N}$
    - $j \in \mathbf{N}$
    - $(i, j) \in \mathbf{E}$

Sample Graphs 1.

- “Lives Near” Graph

nodes

	Actor	Lives near:
n_1	Allison	Ross, Sarah
n_2	Drew	Eliot
n_3	Eliot	Drew
n_4	Keith	Ross, Sarah
n_5	Ross	Allison, Keith, Sarah
n_6	Sarah	Allison, Keith, Ross

Links or edges

$l_1 = (n_1, n_5)$

$l_2 = (n_1, n_6)$

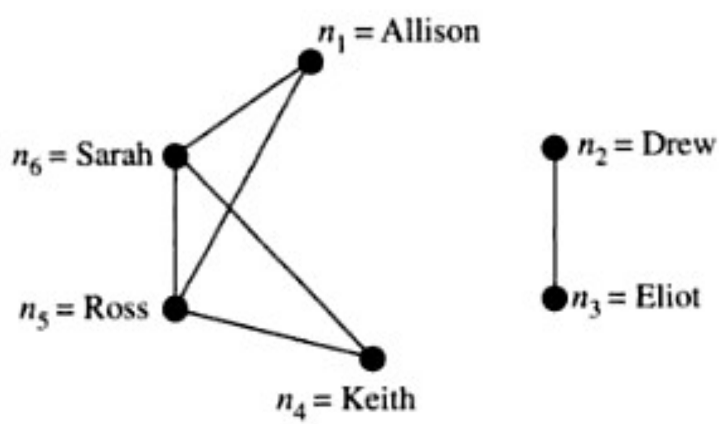
$l_3 = (n_2, n_3)$

$l_4 = (n_4, n_5)$

$l_5 = (n_4, n_6)$

$l_6 = (n_5, n_6)$

Graph



Source: Social network analysis: Methods and applications. Wasserman, Stanley. Cambridge university press, 1994.

Node Degree *d*(*i*)

- Given Node *i*, its degree *d*(*i*) is:
  - the number nodes adjacent to it.

	Actor	Lives near:	Degree
n_1	Allison	Ross, Sarah	2
n_2	Drew	Eliot	1
n_3	Eliot	Drew	1
n_4	Keith	Ross, Sarah	2
n_5	Ross	Allison, Keith, Sarah	3
n_6	Sarah	Allison, Keith, Ross	3

$l_1 = (n_1, n_5)$

$l_2 = (n_1, n_6)$

$l_3 = (n_2, n_3)$

$l_4 = (n_4, n_5)$

$l_5 = (n_4, n_6)$

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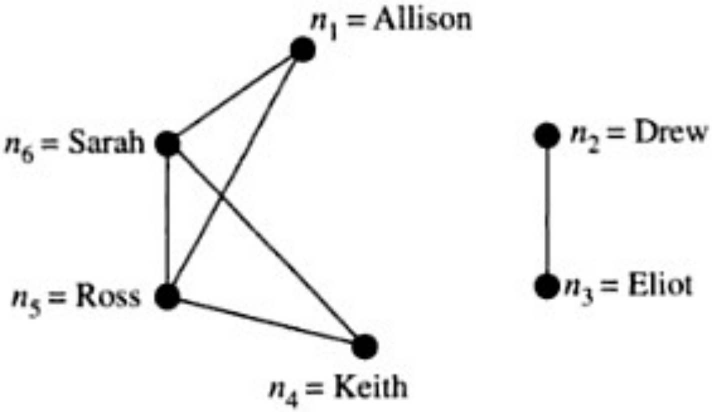
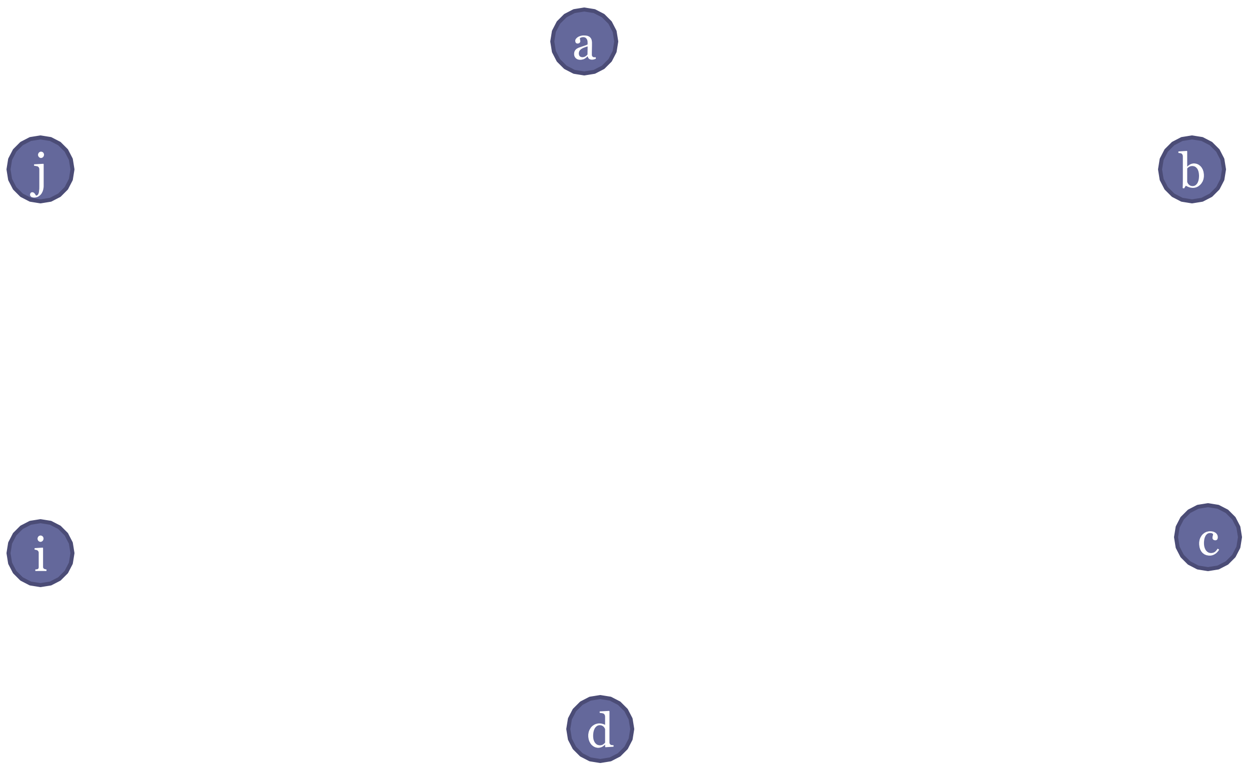


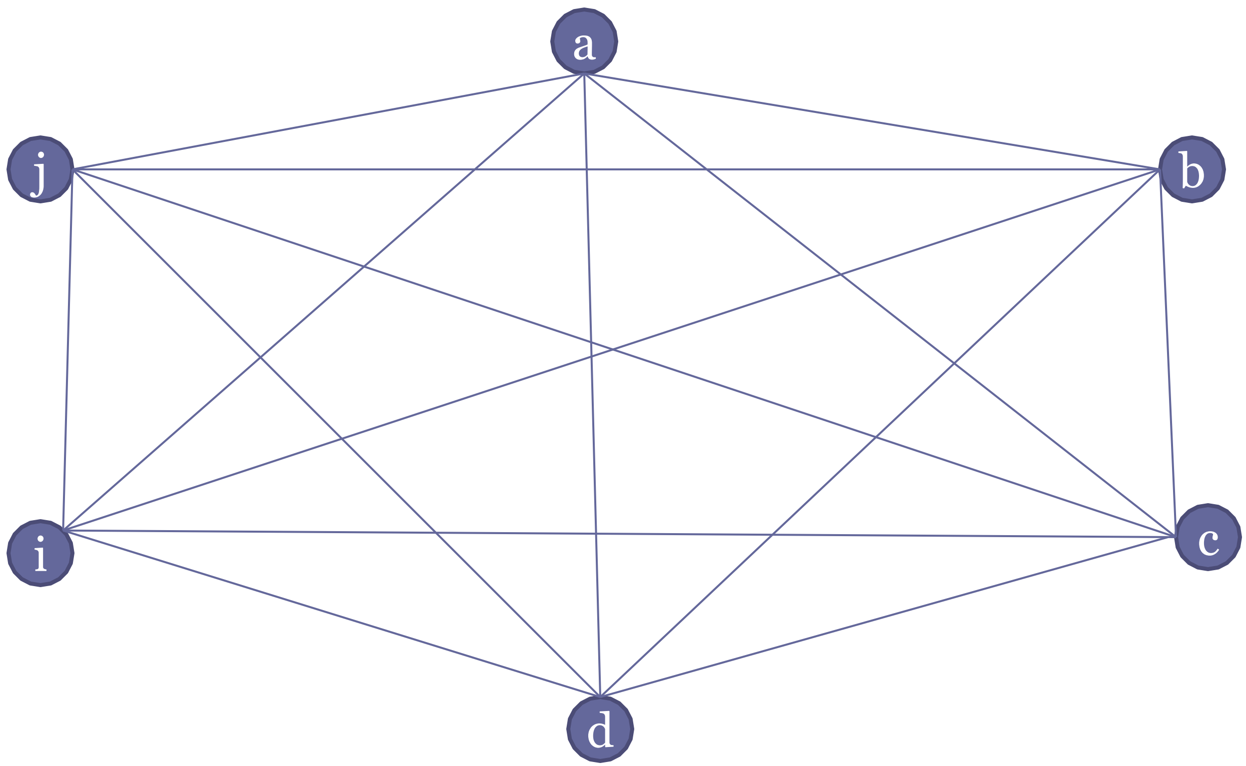
Graph Density

- How many edges are possible?



Graph Density- Cnt.

- $(N-1) + (N-2) + (N-3) + \ldots + 1 = N * (N-1) / 2$



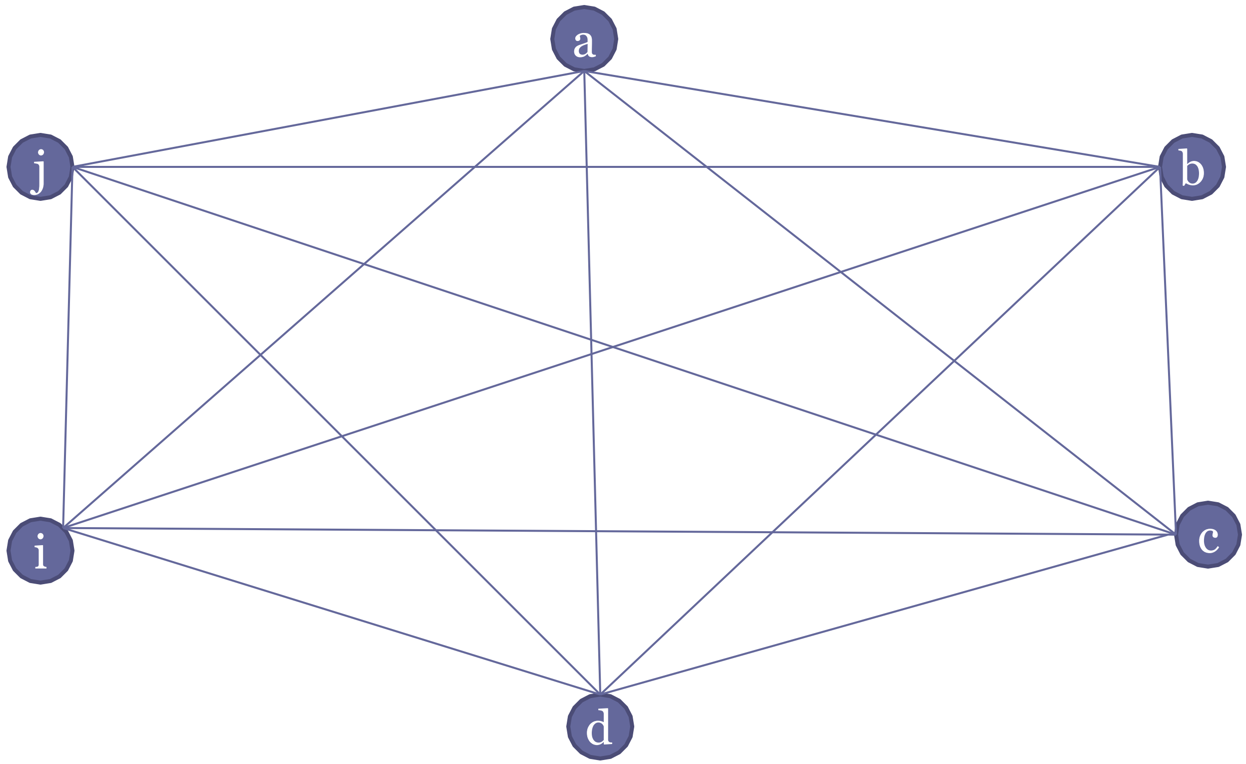
Graph Density- Cnt.

- Graph Density of a given graph G is determined by:
  - the proportion of all possible edges that are present in the graph.
  - with N nodes and E edges, graph density is:

$$\text{Density} = 2 * E / N * (N-1)$$

Complete Graph

- If all edges are present, then all nodes are adjacent (neighbors), and the graph is a *Complete Graph*.



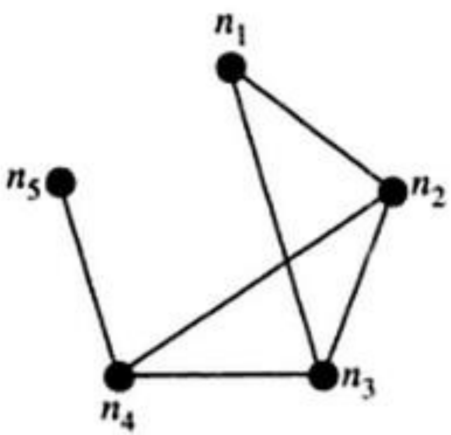
What is the density of a complete graph?

Distance and Diameter

- Distance btw node i and j: $d(i,j)$
  - length of the ***<u>shortest path</u>*** between i and j
- Diameter of a graph
  - the maximum value of $d(i,j)$ for all i and j

The path with min number of edges.

Distance and Diameter- Cnt.



distance

$d(1,2) = 1$

$d(1,3) = 1$

$d(1,4) = 2$

$d(1,5) = 3$

$d(2,3) = 1$

$d(2,4) = 1$

$d(2,5) = 2$

$d(3,4) = 1$

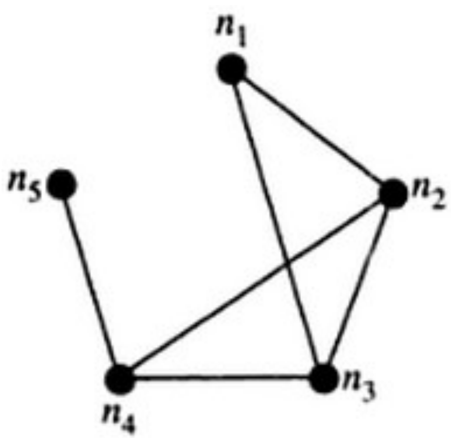
$d(3,5) = 2$

$d(4,5) = 1$

Diameter of graph $= \max d(i,j) = d(1,5) = 3$

What is the distance and diameter of a complete graph?

Adjacency Matrix



$$A = \begin{array}{c|c} & \begin{array}{ccccc} n_1 & n_2 & n_3 & n_4 & n_5 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

- Each row or column represents a node!

$A = A^T$

Properties of adjacency matrix → next session

Graph Connectivity

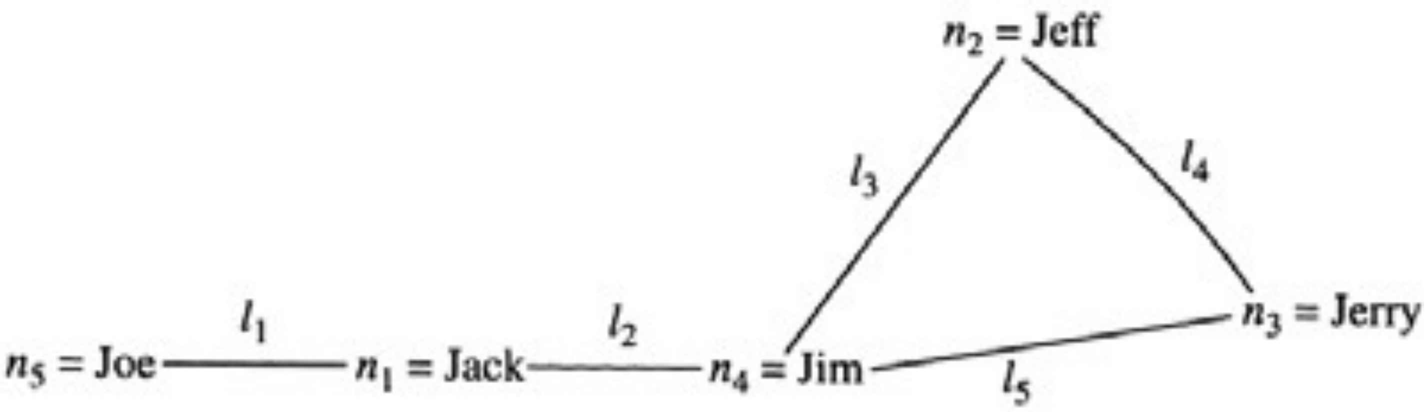
- Indirect connections between nodes:
  - Walks
  - Trails
  - Paths

Graph Connectivity- Cnt.

- Walk
  - A sequence of nodes and edges that starts and ends with nodes where each node is incident to the edges following and preceding it.
- Trail
  - A trail is a walk with distinct edges
- Path
  - A path is a walk with distinct nodes & edges.
- The length of a walk, trail, or path is the number of edges in it.

Graph Connectivity- Cnt.

- Walk
  - A sequence of nodes and edges that starts and ends with nodes where each node is incident to the edges following and preceding it.



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Graph Connectivity- Cnt.

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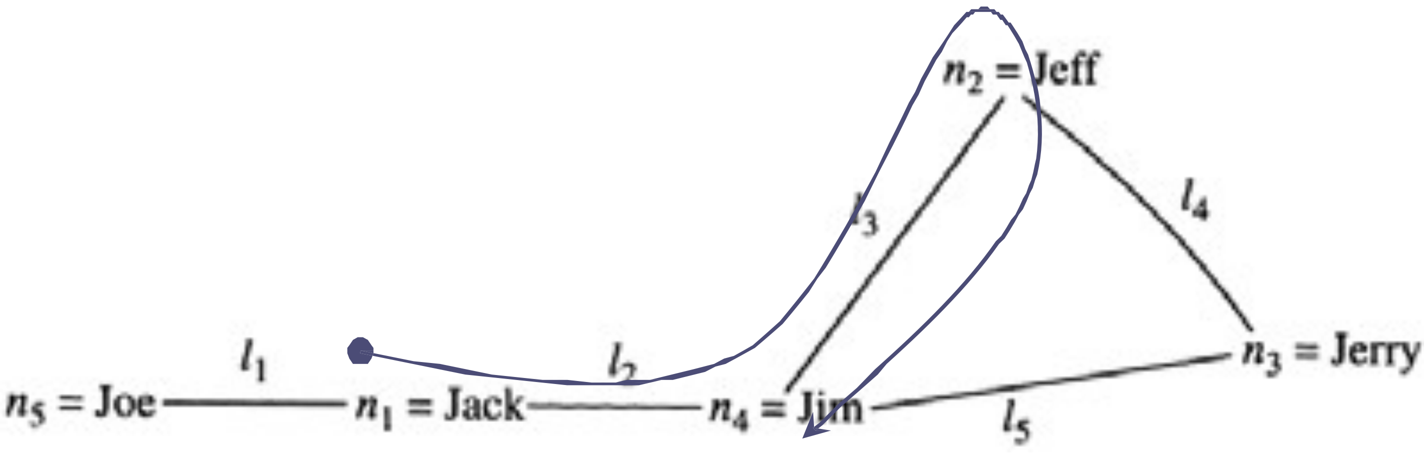
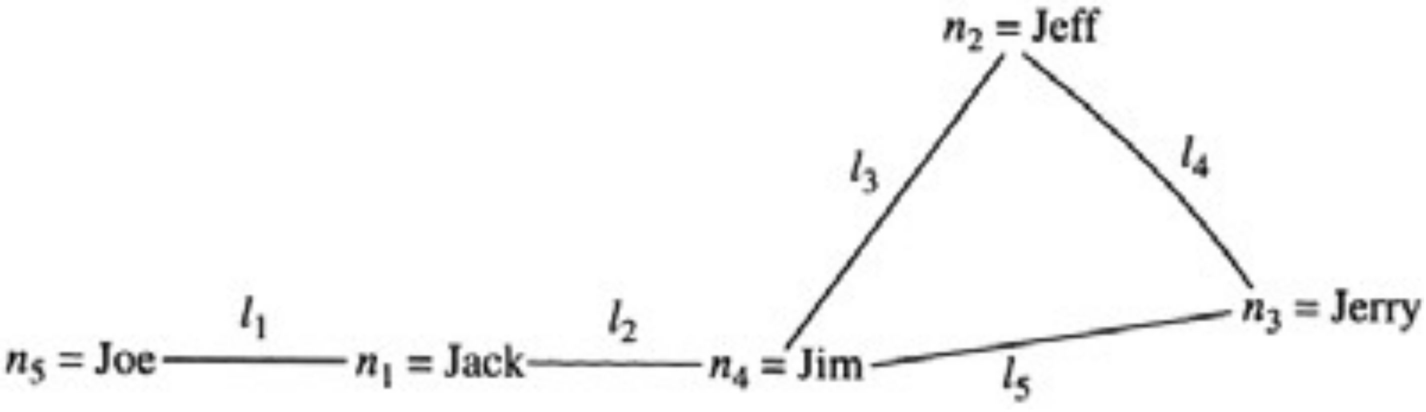


Sample Walk:

$W = n_1\, l_2\, n_4\, l_3\, n_2\, l_3\ \ n_4$

Graph Connectivity- Cnt.

- Trail
  - A trail is a walk in which all edges are distinct, although some node(s) may be included more than once.



Graph Connectivity- Cnt.

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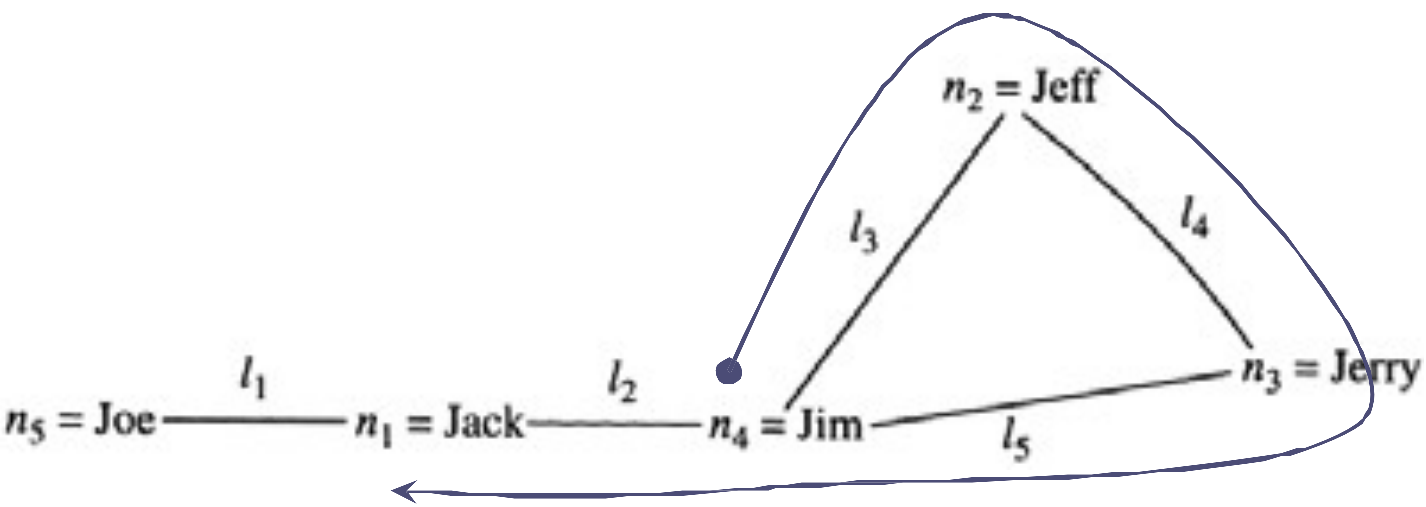
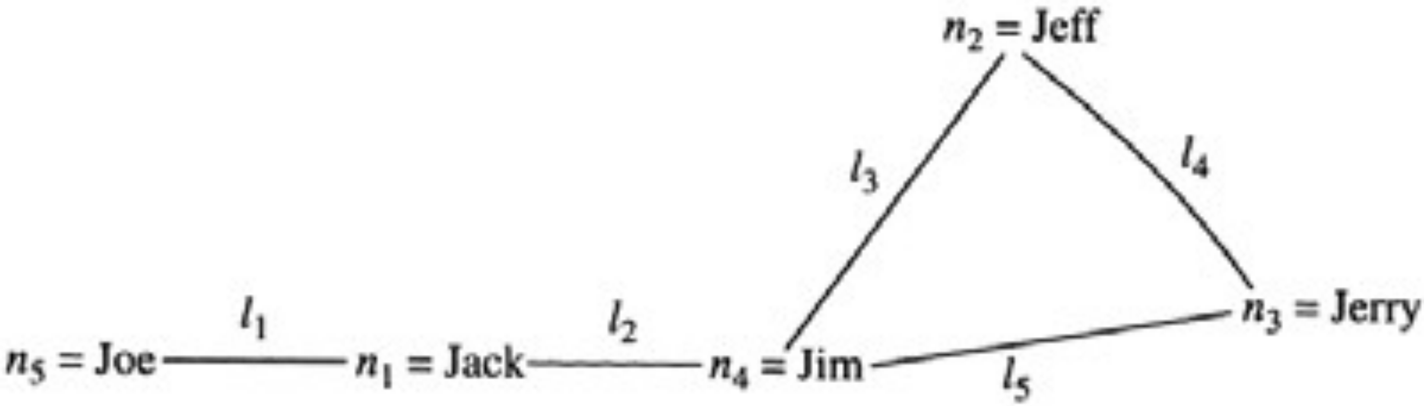


Sample Trail:

$T = n_4\, l_3\, n_2\, l_4\, n_3\, l_5\, n_4\, l_2\, n_1$

Graph Connectivity- Cnt.

- Path
  - A path is a walk in which all nodes and all edges are distinct.



Graph Connectivity- Cnt.

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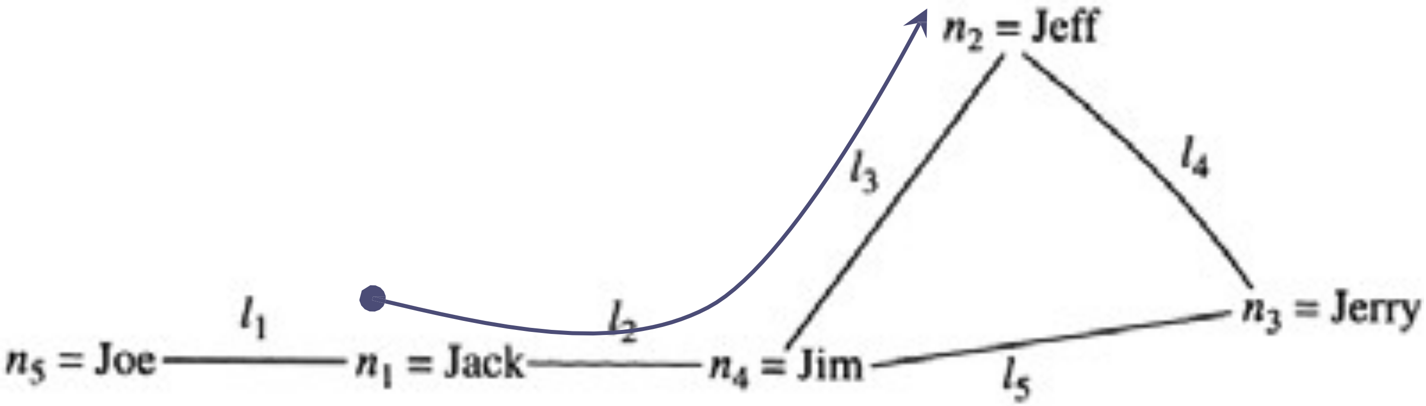
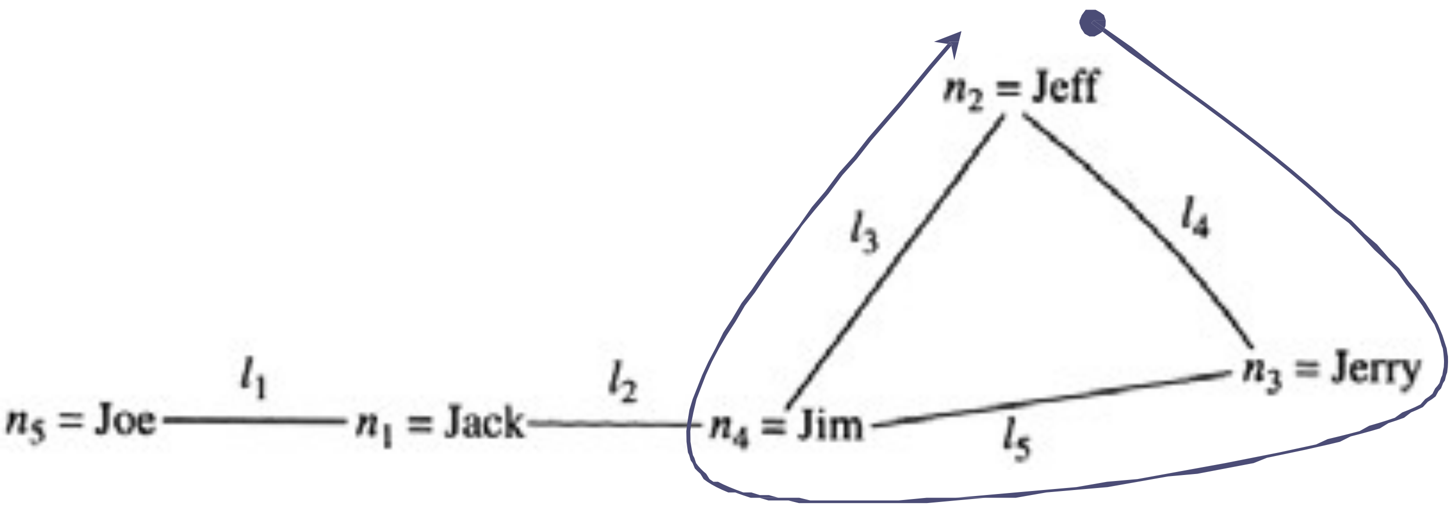


Sample Path:

$P = n_1 l_2 n_4 l_3 \ n_2$

Graph Connectivity- Cnt.

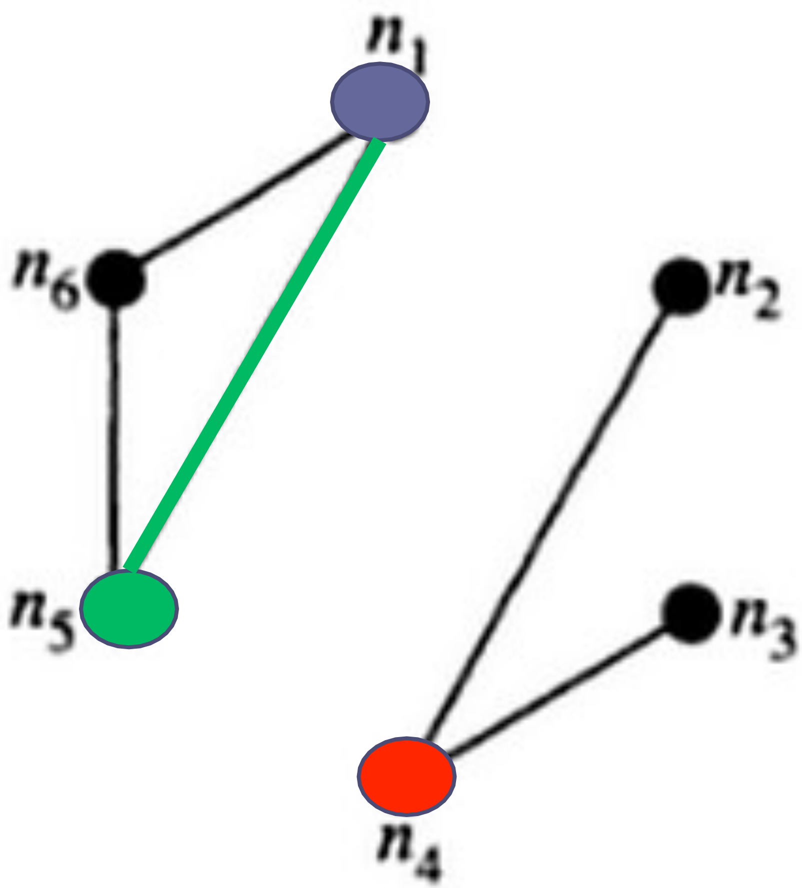
- Is this a Walk? Trail? Path?
  - We call a *closed path* is a Cycle!



$n_2\, l_4\, n_3\, l_5\, n_4\, l_3\;\; n_2$

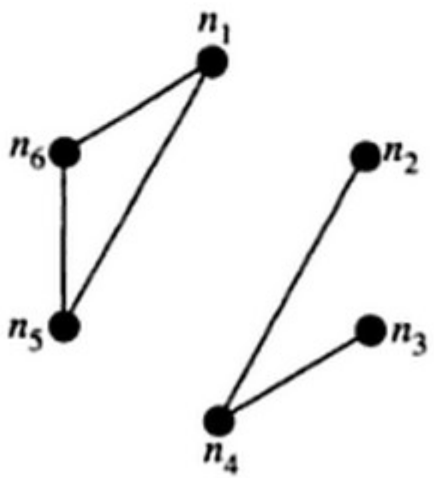
Reachability

- If there is a **path between nodes** *i* and *j*, then *i* and *j* are reachable from each other.



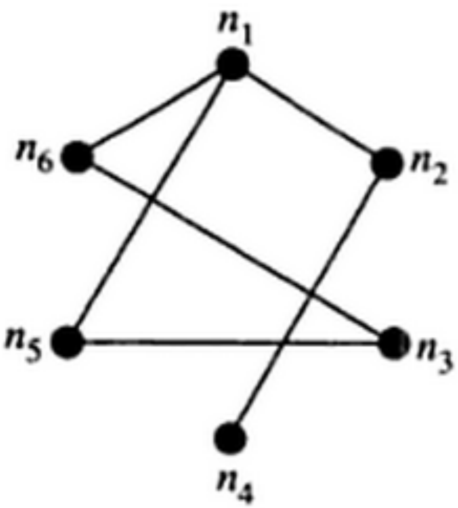
Connected Graph

- A graph is connected if ***every pair of its nodes*** are reachable from each other
  - i.e. there is a path between them.



Disconnected Graph

How can we make this graph connected?



Connected Graph

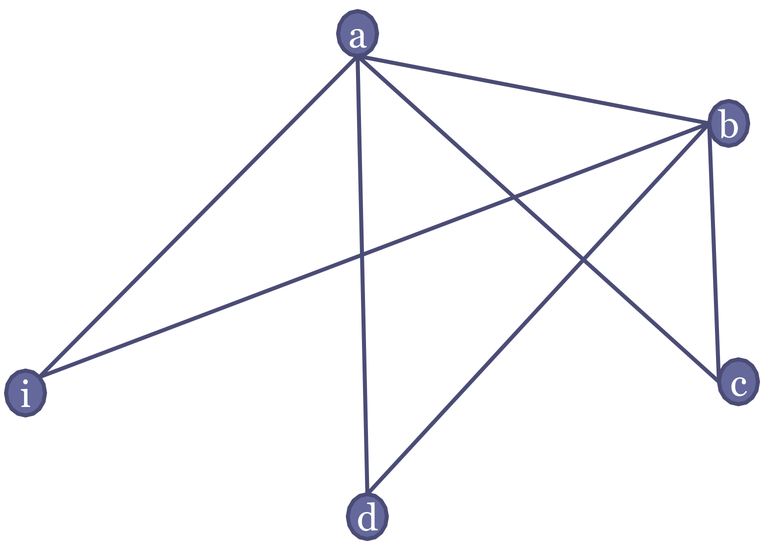
and this graph disconnected?

Sub-graphs

- Graph G_s is a sub-graph of G if its nodes and edges are a subset of G's nodes and edges respectively.

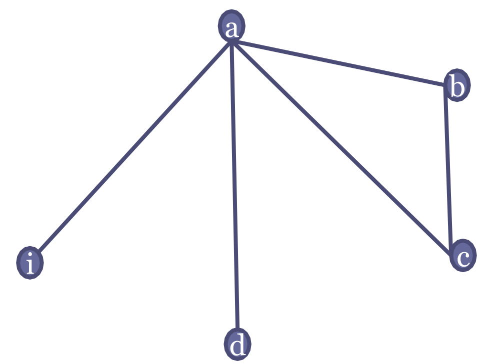
Sub-graphs- Cnt.

- Graph G_s is a sub-graph of G if its nodes and edges are a subset nodes and edges of G respectively.

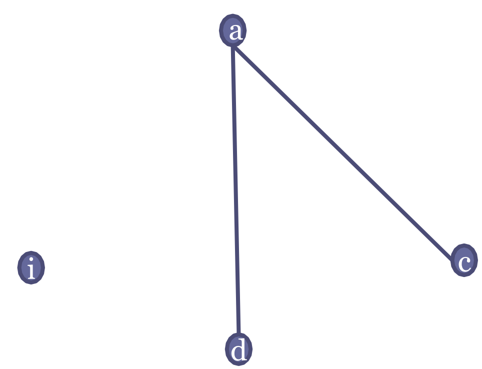


G

G_{s1}



G_{s2}

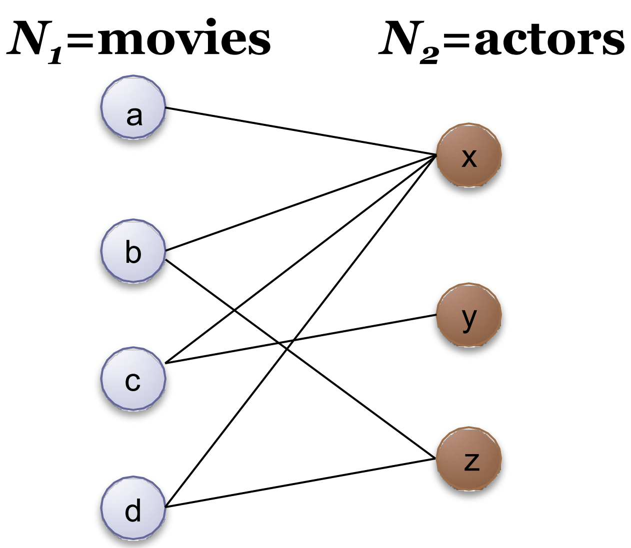


Graph Types

- Several types of graphs:
  - Bipartite graphs
  - Digraphs
  - Multigraphs
  - Hypergraphs
  - Weighted/Signed

Graph Types- Bipartite Graphs

- A bipartite graph is an undirected graph in which
  - nodes can be partitioned into two (disjoint) sets N_1 and N_2 such that:
    - $(u, v) \in E$ implies either $u \in N_1$ and $v \in N_2$ or vice versa
  - So, all edges go between the two sets N_1 and N_2 but not within N_1 or N_2.



N_1={a,b,c,d}

N_2={x,y,z}

Graph Types- Digraphs

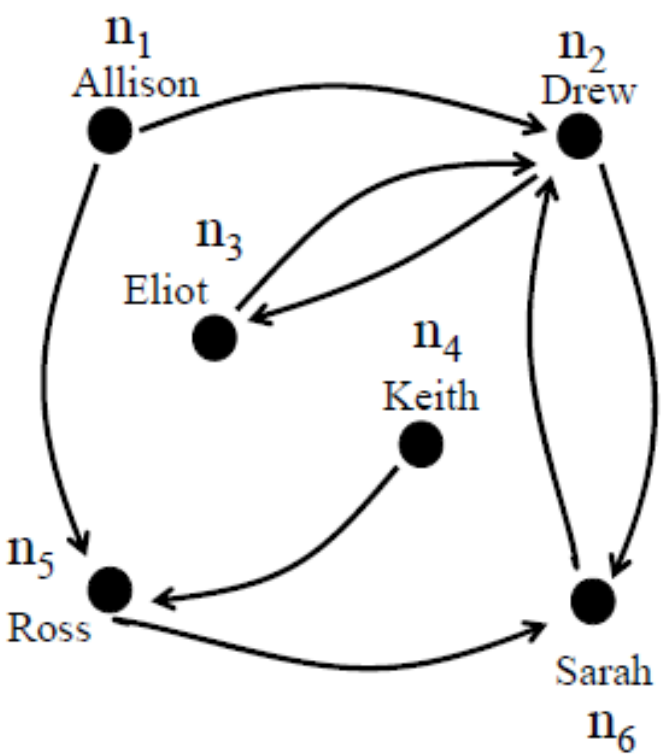
- Digraphs or Directed Graphs
  - Edges are directed
- Adjacency:
  - There is a direct edge btw nodes!
    - $i \in N$
    - $j \in N$
    - $(i, j) \in E$

Graph Types- Digraphs- Cnt.

- Node Indegree and Outdegree
  - Indegree
    - The indegree of a node, $d_I(i)$, is the number of nodes that link to *i*,
  - Outdegree
    - The outdegree of a node, $d_O(i)$, is the number of nodes that are linked by *i*,

- Indegree: number of edges terminating at *i*.
- Outdegree: number of edges originating at *i*.

Graph Types- Digraphs- Cnt.



$$d_O(n_i) = \sum_{j=1}^{n} A_{ij}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{|c|} \hline 2 \\ \hline 2 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$$

$$d_I(n_j) = \sum_{i=1}^{n} A_{ij}$$

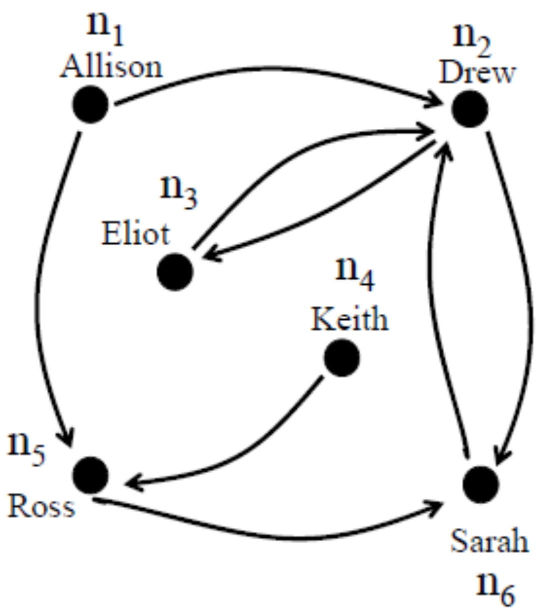
0	3	1	0	2	2

$A \mathrel{!}= A^T$

Graph Types- Digraphs- Cnt.

- Density of Digraph:
  - Number of all possible edges in Digraph?
    - N * (N-1)

$$\frac{E}{N * (N-1)}$$



Graph Types- Digraphs- Cnt.

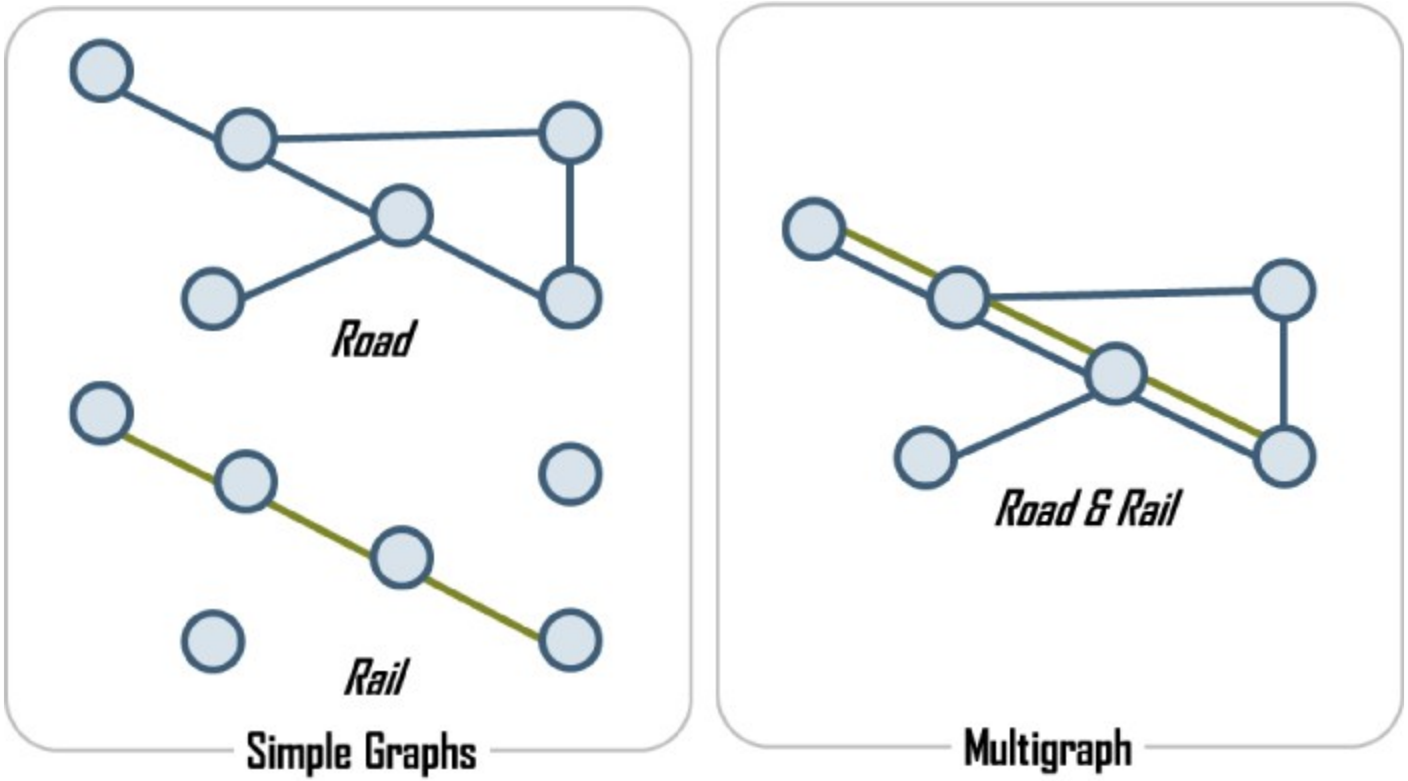
- Connectivity
  - Walks
  - Trails
  - Paths

- The same as before just links are directed!

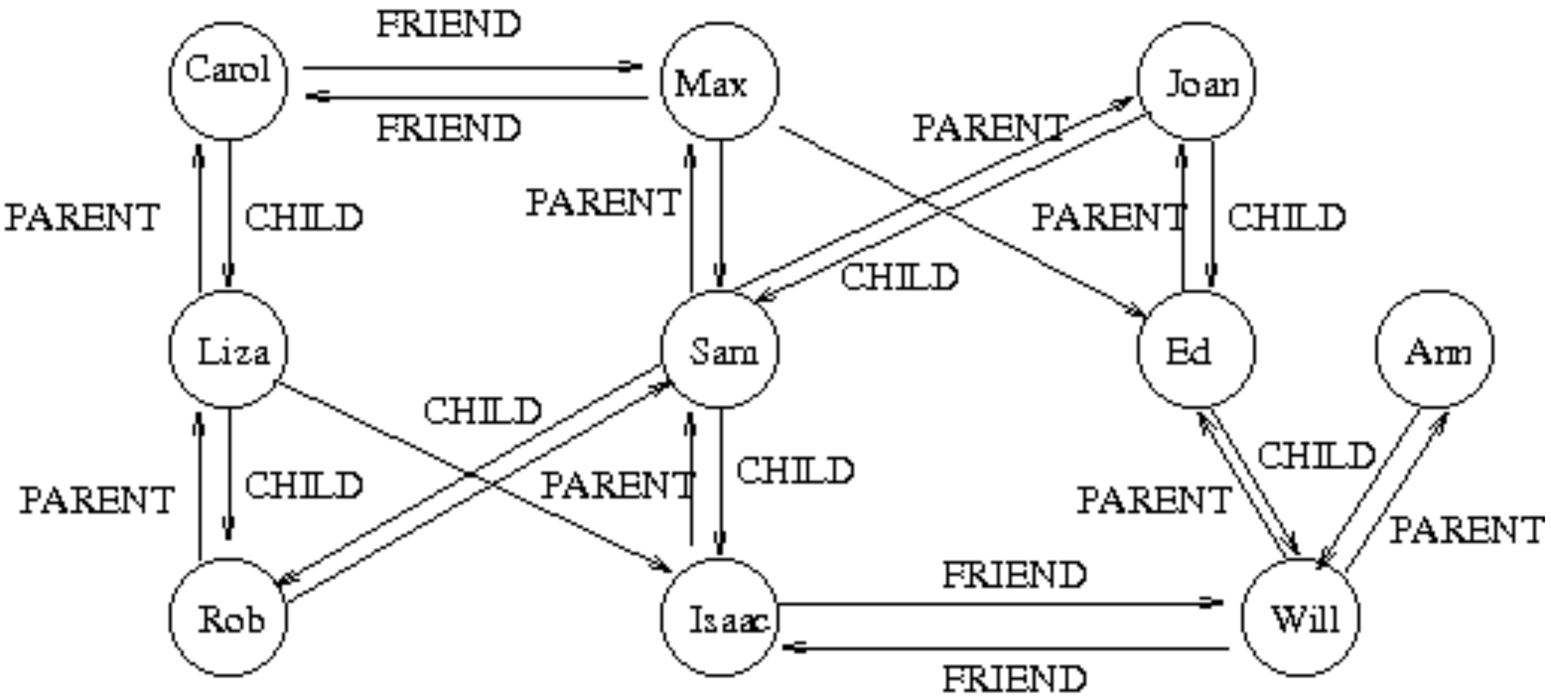
Graph Types- Multigraphs

- A Multigraph (or multivariate graph) *G* consists of:
  - a set of nodes, *and*
  - two or more sets of edges, $E^{+} = \{\boldsymbol{E_1}, \boldsymbol{E_2}, \ldots, \boldsymbol{E_r}\}$, r is the number of edge sets.

Multigraph 1.



Multigraph 2.



Graph Types- Multigraphs- Cnt.

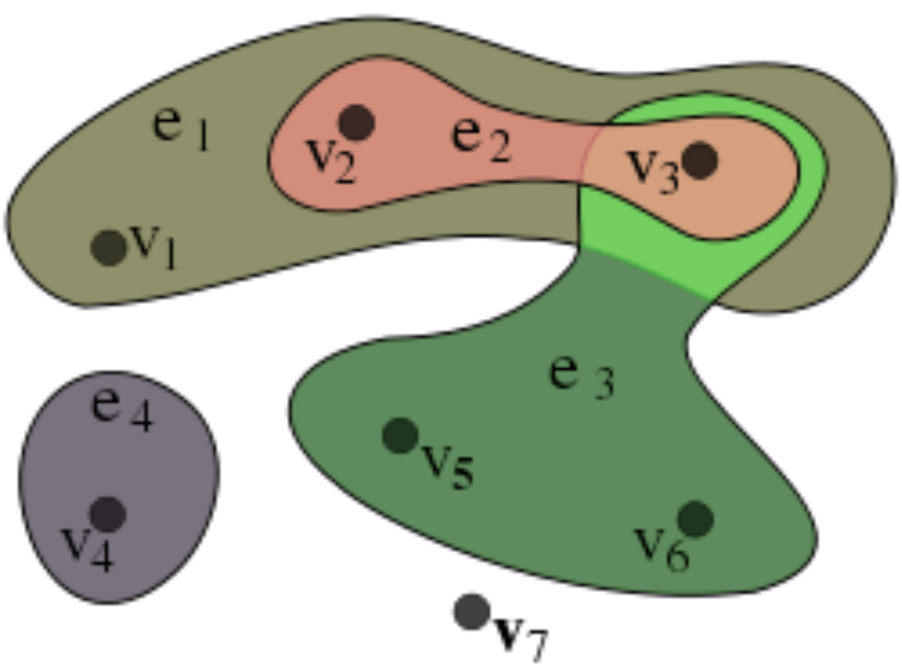
- Number of edges btw any two nodes in a multigraph?
  - $E^+ = \{\boldsymbol{E_1}, \boldsymbol{E_2}, \ldots, \boldsymbol{E_r}\}$, r is the number of sets of edges
    - Undirected multigraph
      - [0, r]
    - Directed multigraph
      - [0, 2*r]

Graph Types- Hypergraphs

- A hypergraph is a graph in which an edge can connect any number of nodes.
- In a hypergraph, $\boldsymbol{E}$ is a set of non-empty subsets of $\boldsymbol{N}$ called *hyperedges*.

Graph Types- Hypergraphs- Cnt.

- A hypergraph is a graph in which an edge can connect any number of nodes.
- In a hypergraph, ***E*** is a set of non-empty subsets of ***N*** called *hyperedges*.



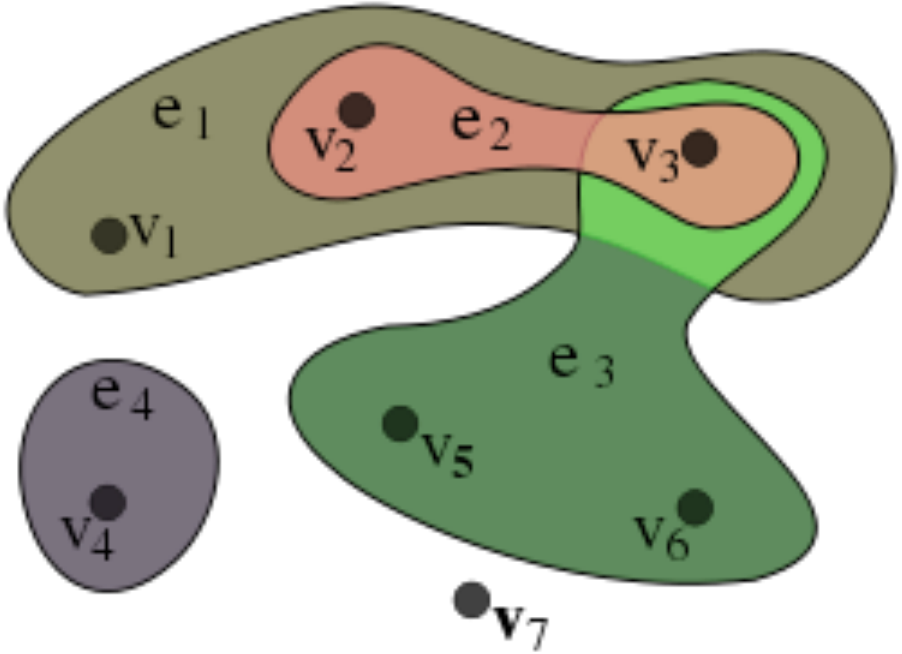
N$=\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

E$=\{e_1, e_2, e_3, e_4\}=$

$\{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\}$

Graph Types- Hypergraphs- Cnt.

- Applications:
  - Recom. systems (communities as edges),
  - Image retrieval (correlations as edges),
  - Bioinformatics (interactions or semantic types as edges).



N$=\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

E$=\{e_1, e_2, e_3, e_4\}=$

$\{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\}$

Weighted/Signed Graphs

- Edges may carry additional information
  - Tie strength → how good are two nodes as friends?
  - Distance → how long is the distance btw two cities?
  - Delay → how long does the transmission take btw two cities?
  - Signs → two nodes are friends or enemies?

Reading

- Ch. 22 Elementary Graph Algorithms [CLRS]

Network Basics 2

Graph ML

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Lecture Topics

- Connected Components
- Breadth-First Search
- Depth-First Search
- Shortest Path Algorithm
  - Dijkstra’s algorithm

Connected Components

- Connected component of a graph is a subset of nodes such that:
  - every node in the subset has a path to every other; and
  - the subset is not part of a bigger component.

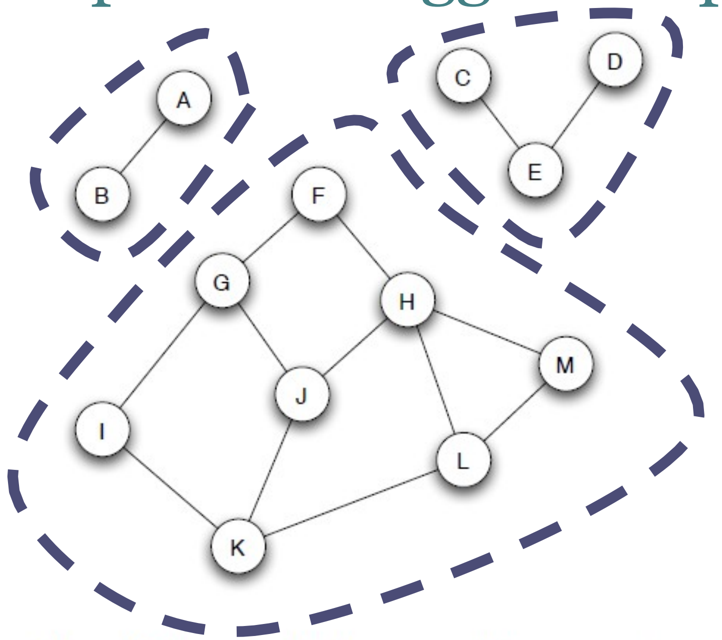


Figure 2.5: A graph with three connected components.

Connected Components

- Connected component of a graph is a subset of nodes such that:
  - every node in the subset has a path to every other; and
  - the subset is not part of a bigger component.

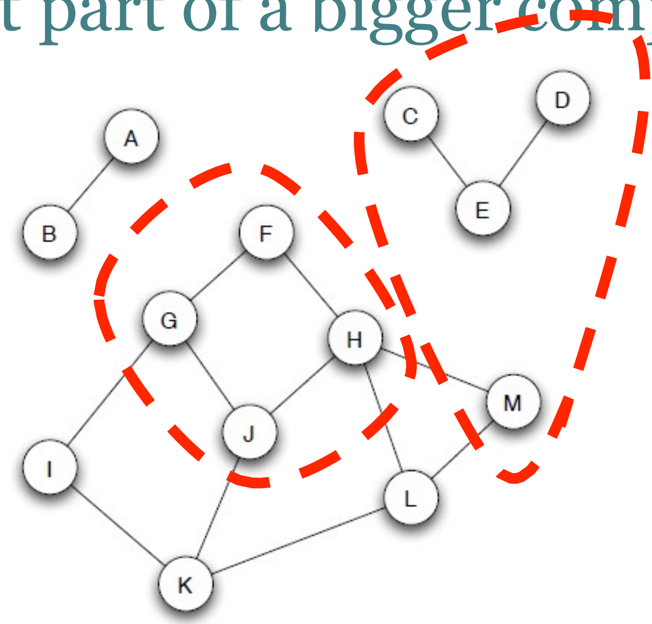
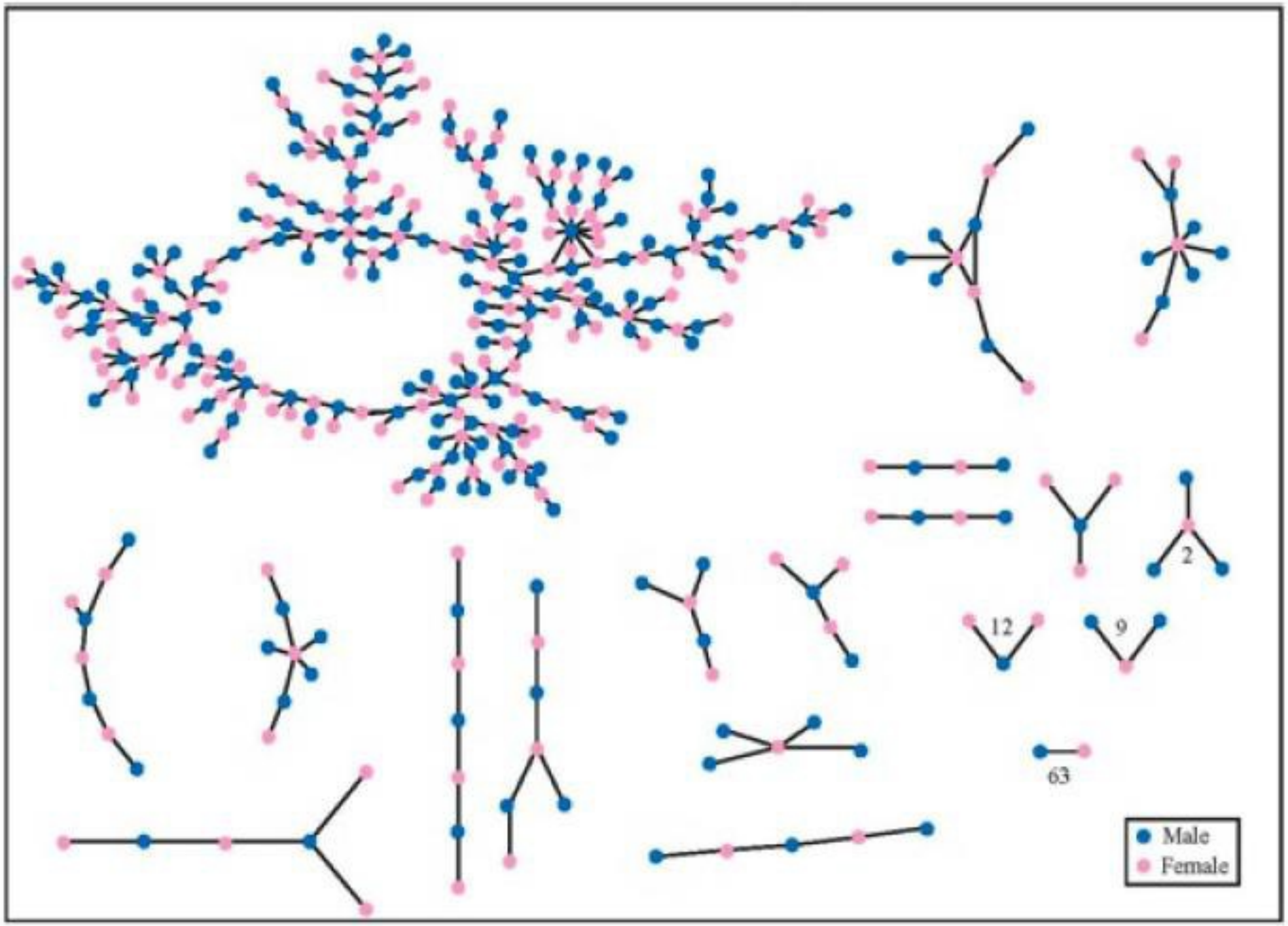


Figure 2.5: A graph with three connected components.

Connected Components- Cnt.



Connected Components- Cnt.

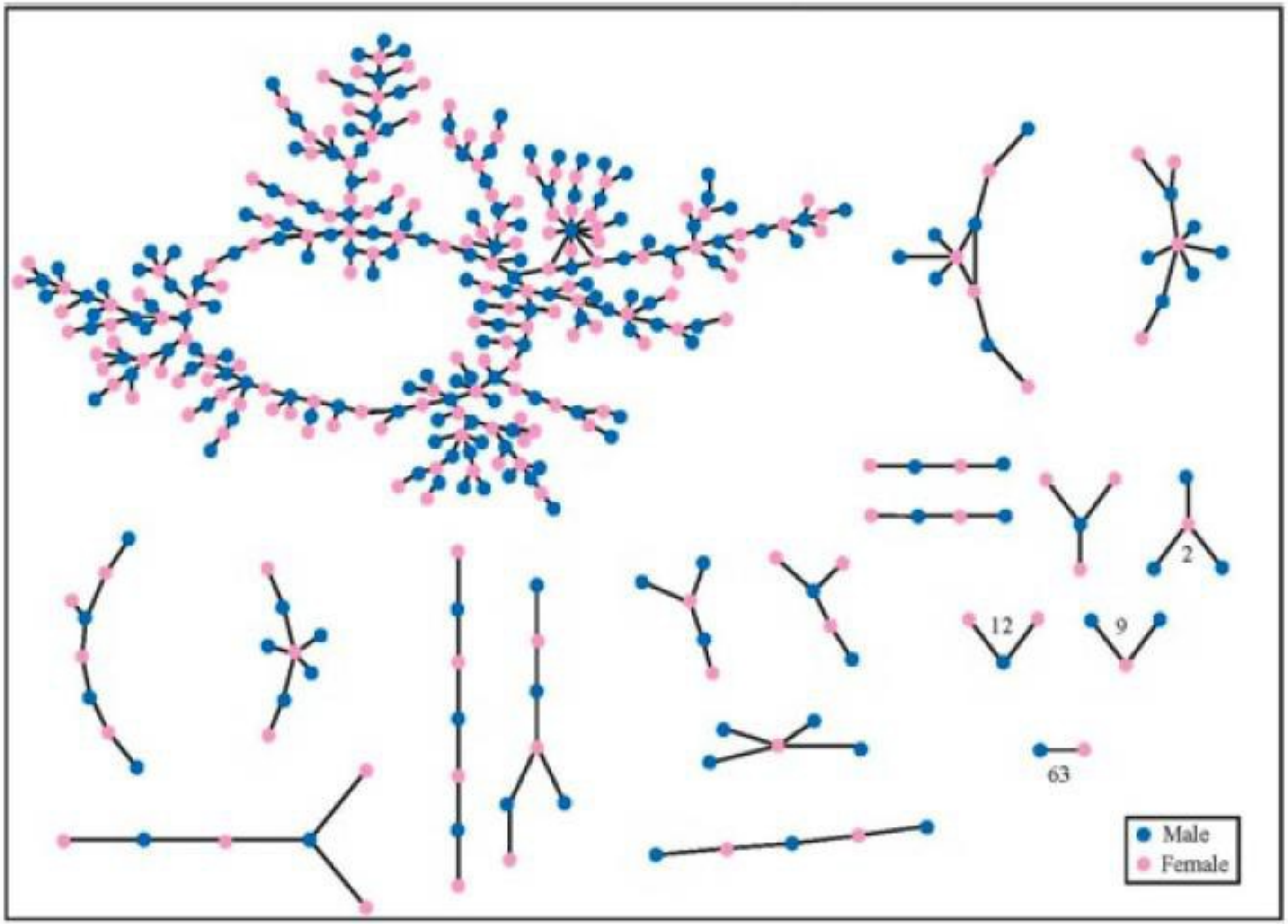


Figure 2.7: A network in which the nodes are students in a large American high school, and an edge joins two who had a romantic relationship at some point during the 18-month period in which the study was conducted [49].

Breadth & Depth-First Search

- General techniques for traversing graphs!
  - Start from a given node *s* (i.e. start node) and visit all nodes and edges in the graph.
- Compute the connected components of graph!
  - Use components to determine whether graph is connected!
    - How?
  - Use components to determine if there is a path btw node pairs!
    - How?

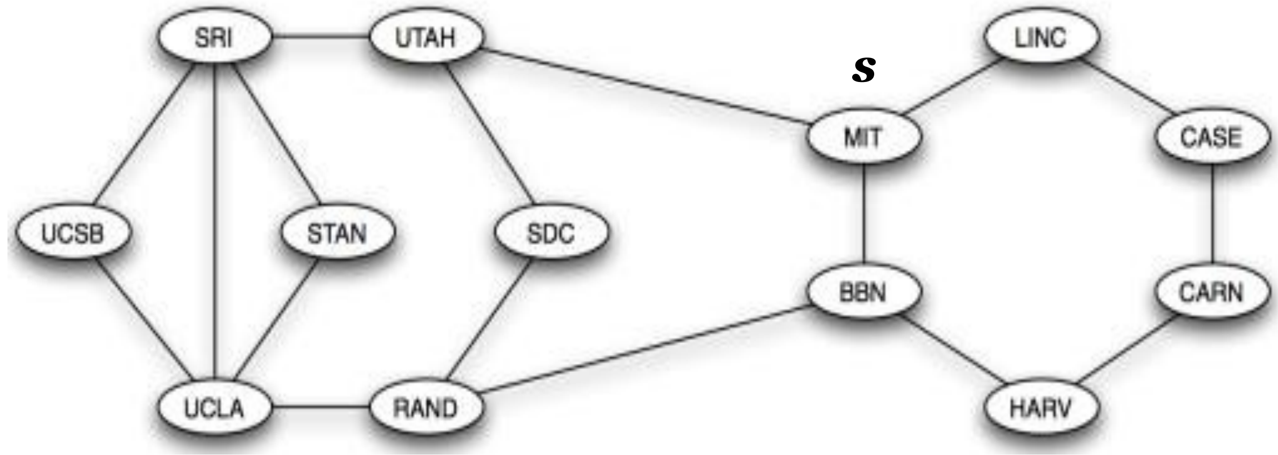
Breadth-First Search

- Start with *s*
- Visit all neighbors of *s*
  - these are called **level-1 nodes**
- Visit all neighbors of level-1 nodes
  - these are called **level-2 nodes**
- Repeat until all nodes are visited.
  - Each Node is only visited once.

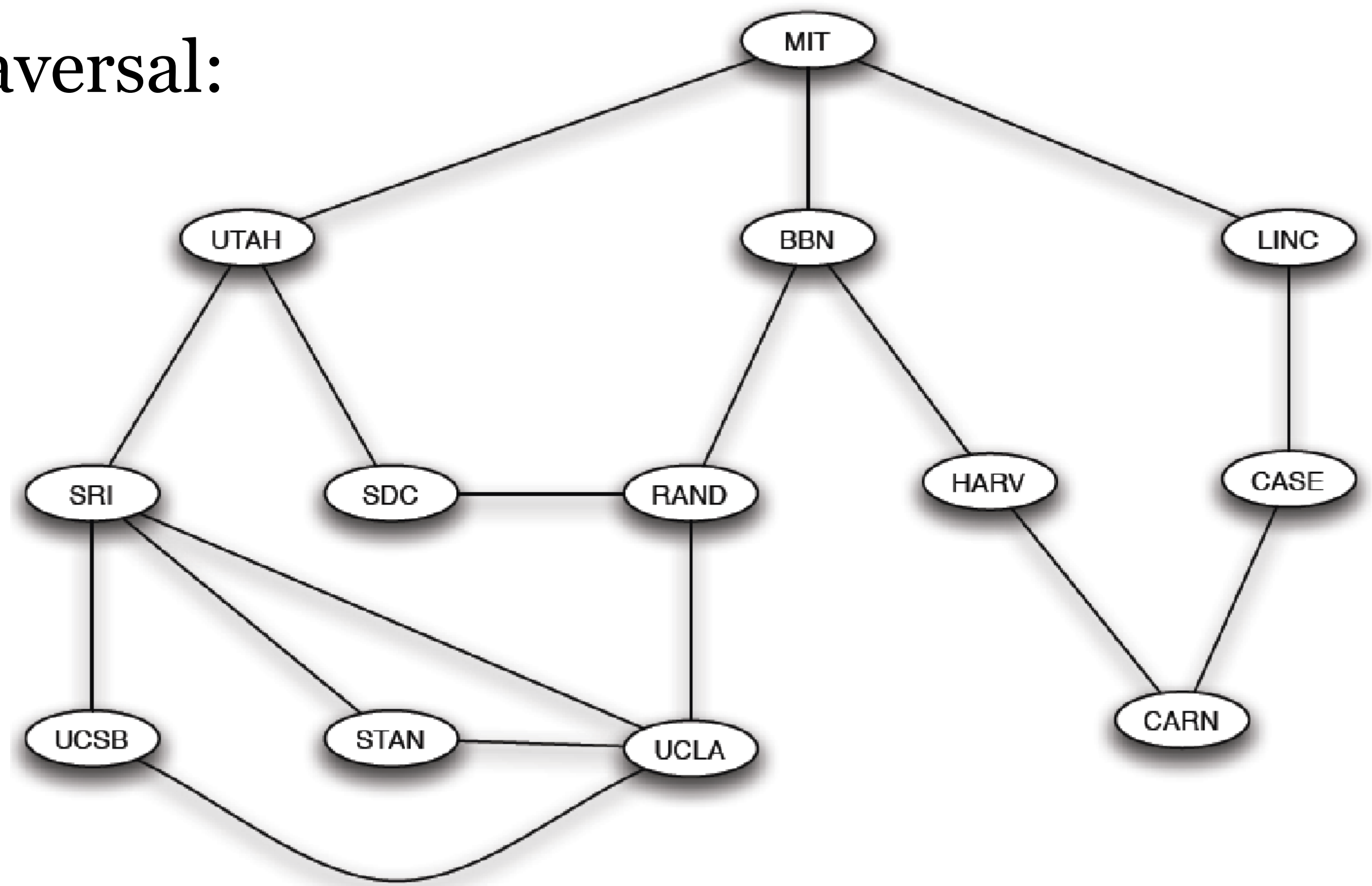
- Key Point:
  - All level-k nodes should be visited before any level-(k+1) node!

Example 1.

- Graph G:

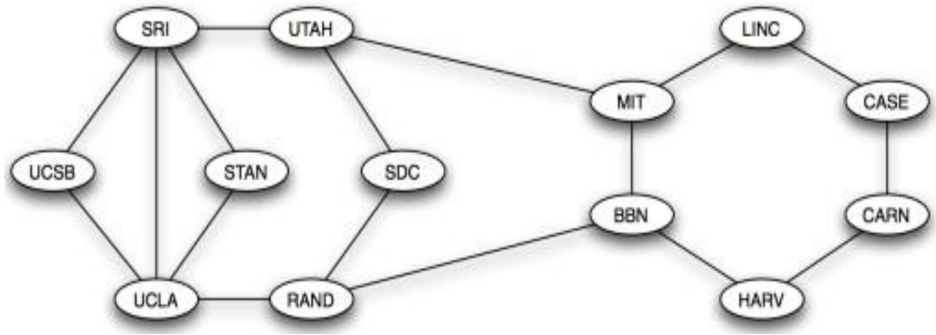


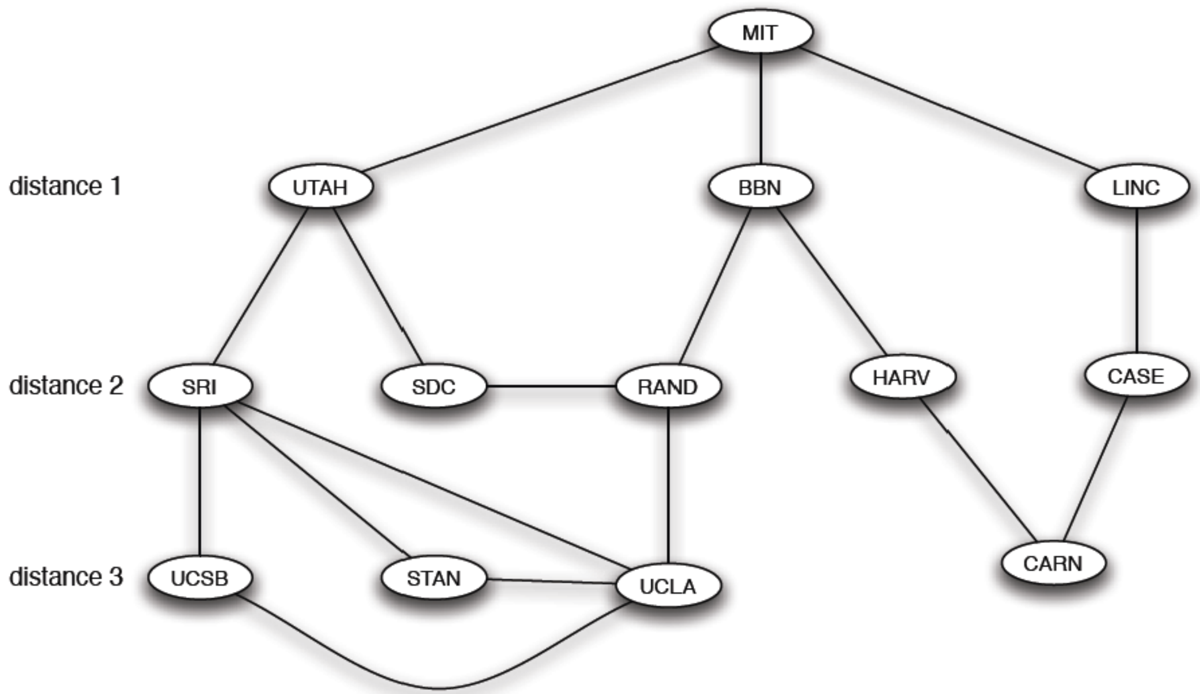
- Its BFS traversal:



Example 1. BFS -Cnt.

- BFS traversal:
  - Distance to root at level-*i*?
  - Components?
    - Connectivity?
    - Paths?





Depth-First Search

- Starts from s
- Explores as far as possible along each branch before backtracking.
    - Visit a neighbor of *s* [say v_1]
    - Visit a neighbor of v_1 [say v_2]
    - Repeat until all nodes are visited.

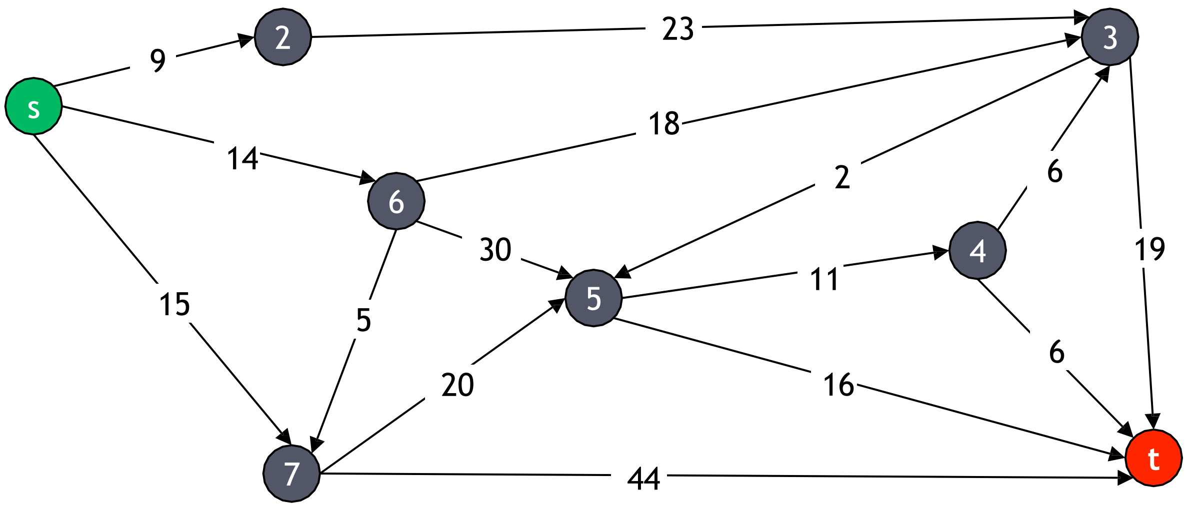
Shortest Path Algorithms

- Given a weighted directed graph and two nodes *s* and *t*, find the shortest path from *s* to *t*.
  - Cost of path = sum of edge weights in path

Shortest Path Algorithms- Cnt.

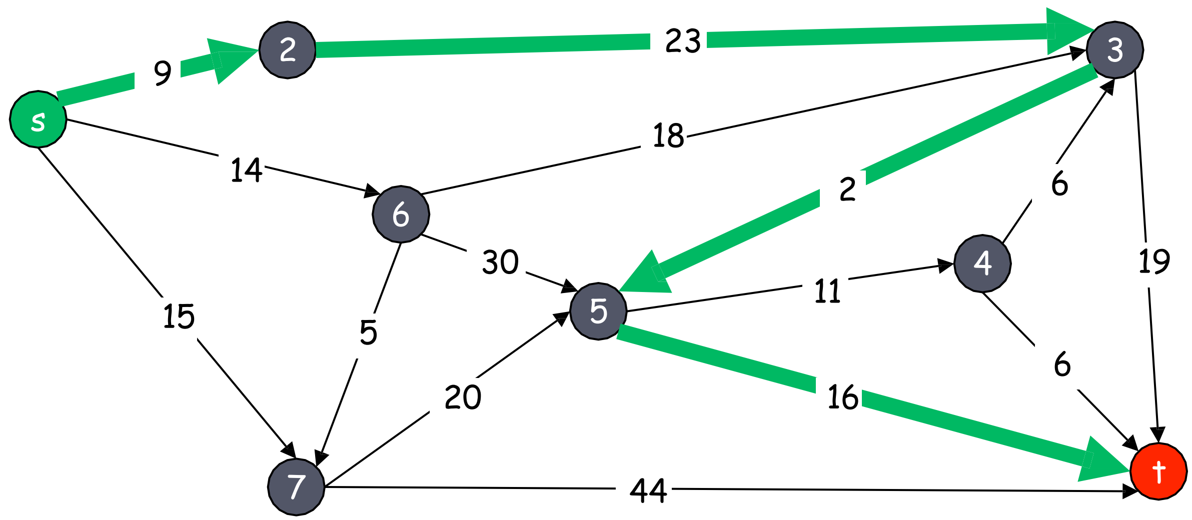
- Dijkstra’s algorithm
- The Bellman-Ford algorithm
- The Floyd-Warshall algorithm
- Johnson's algorithm
- Etc.

Shortest Path Algorithms- Cnt.



- Shortest path from *s* to *t*?

Shortest Path Algorithms- Cnt.



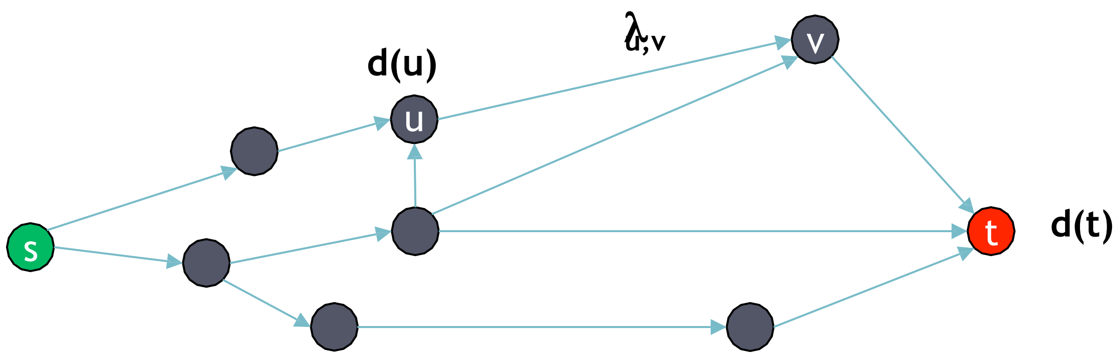
- Shortest Path= s-2-3-5-t
- Cost of path = 9 + 23 + 2 + 16 = 48.

Shortest Path Algorithms- Cnt.

- Applications
  - Small World Phenomenon
  - Internet packet routing
  - Flight reservations
  - Driving directions
  - …

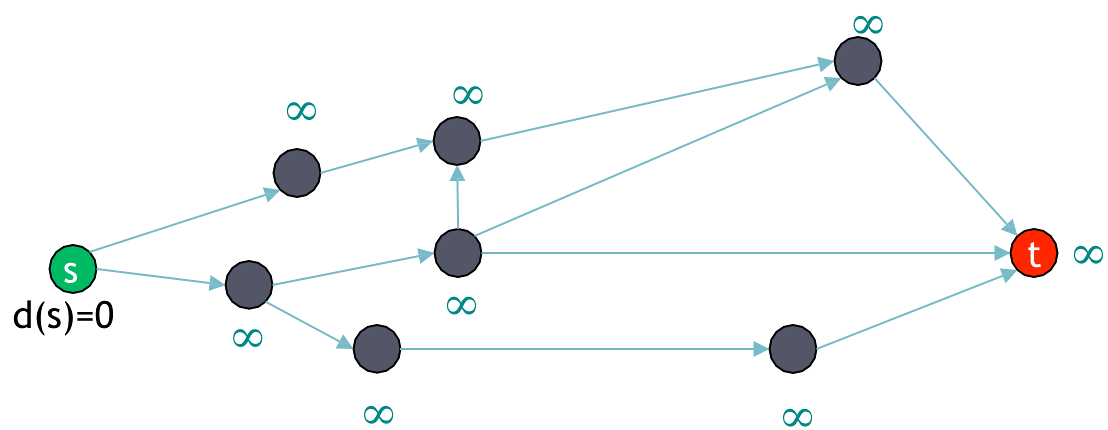
Dijkstra algorithm

- Weighted Directed graph G = (**N**, **E**),
  - s: source node
  - t: target node
  - $l_{(u,v)}$: weight of the edge btw nodes u and v
  - d(u): shortest path distance from s to u.
    - sum of edge weights in path
- We aim to compute d(t)!



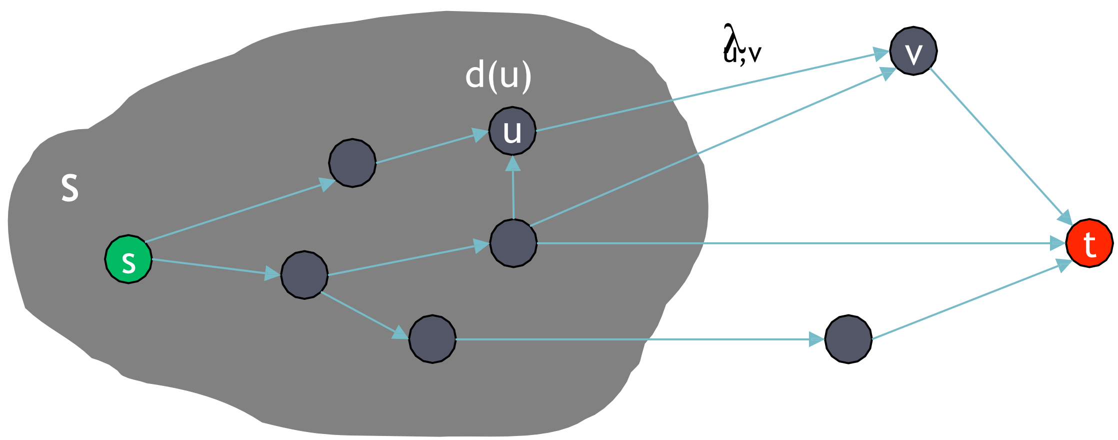
Dijkstra algorithm- Cnt.

- Initialization?
  - d(s) = 0
  - d(u)= ∞ for all other nodes



Dijkstra algorithm- Cnt.

- To find the shortest path from *s* to *t*:
  - Maintain a set of ***explored nodes*** **S** for which we have determined the shortest path distance from s to any u ϵ **S**.
  - Repeatedly expand **S**.

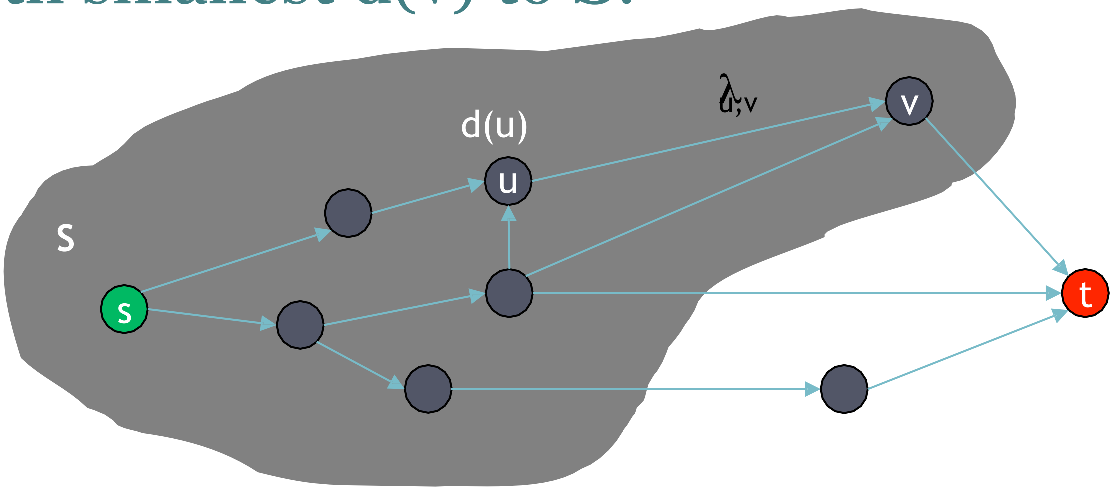


Dijkstra algorithm- Cnt.

- Repeatedly expand **S**?
  - Repeatedly update *d(.)* for the unexplored nodes:

$$\textbf{if } d(v) > d(u) + l_{(u,v)}$$
$$\textbf{then } d(v) \leftarrow \quad d(u) + l_{(u,\mathrm{v})}$$

  - add v with smallest d(v) to **S**.



Dijkstra algorithm- Cnt.

- $d(s) \leftarrow 0$
- **for** each $v \in N - \{s\}$
  - **do** $d(v) \leftarrow \infty$
- $S \leftarrow \emptyset$
- $Q \leftarrow N$ ▷ Q is a set maintaining $N - S$
- **while** $Q \neq \emptyset$
  - do $u \leftarrow$ EXTRACT-MIN(Q)
    - $S \leftarrow S \cup \{u\}$
    - **for** each $v \in Adj(u)$
      - **do if** $d(v) > d(u) + l_{(u,v)}$
        - **then** $d(v) \leftarrow d(u) + l_{(u,v)}$

Set of explored nodes

Set of unexplored nodes

Returns node u ∈ Q that has minimum d(u)

Add it to explored nodes

Update d(.) for all neighbors of u: this is called **relaxation**!

Example 1.

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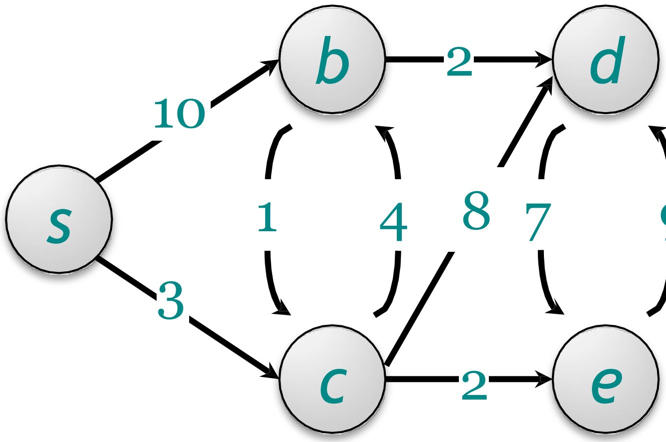

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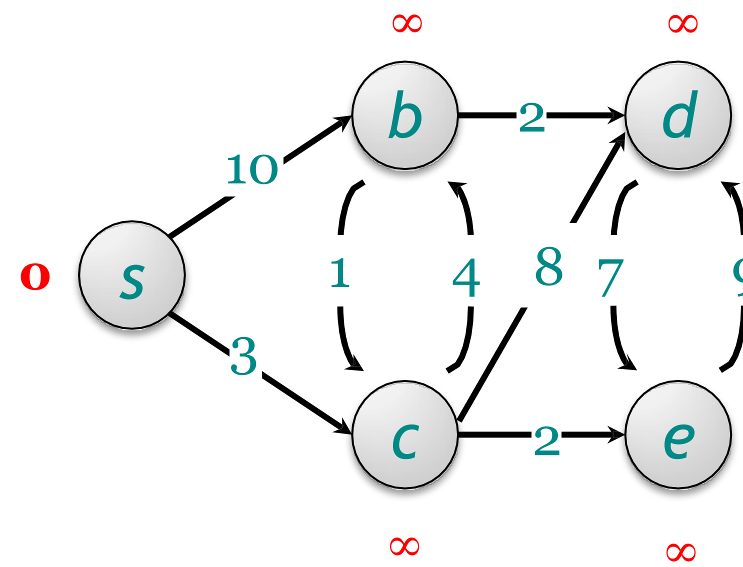

S={}

Q={s, b, c, d, e}

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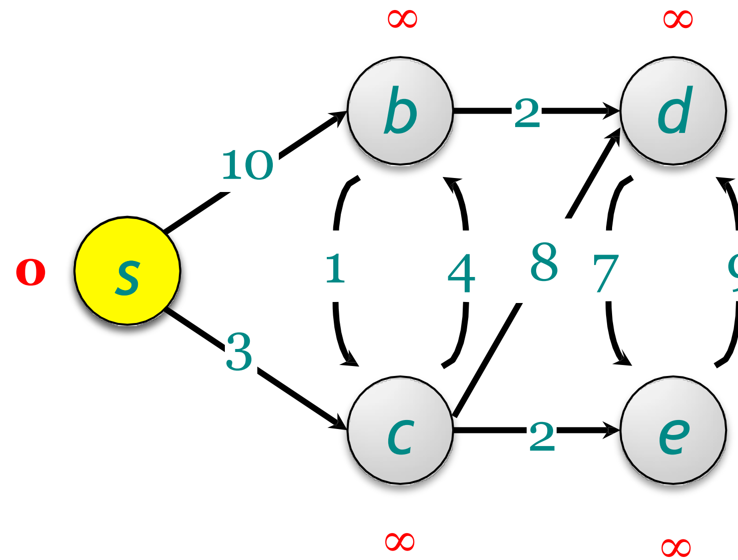

S={}

s

Q={b, c, d, e}

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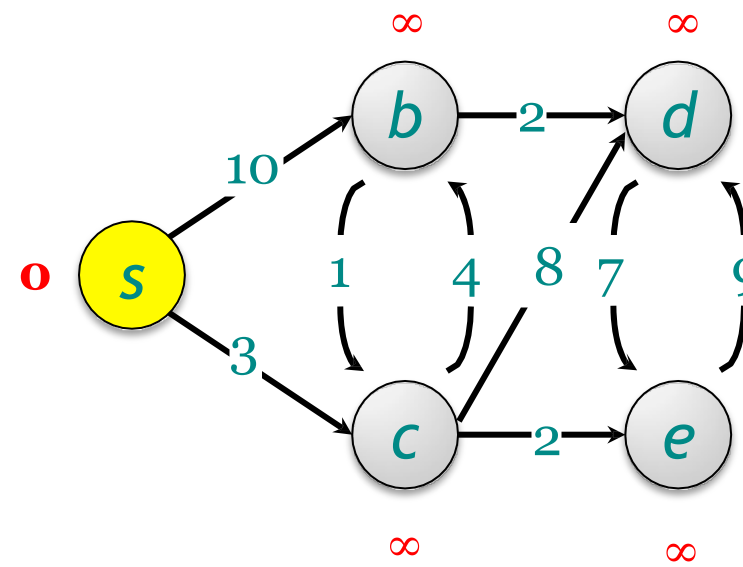

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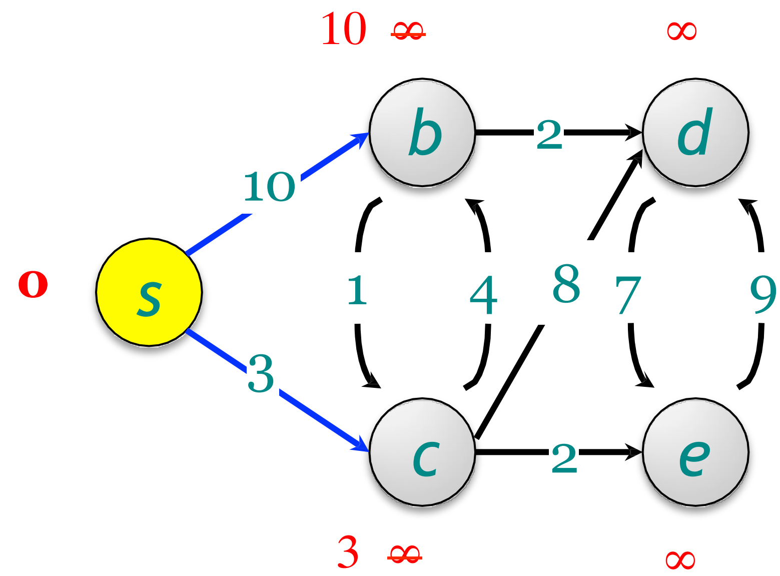

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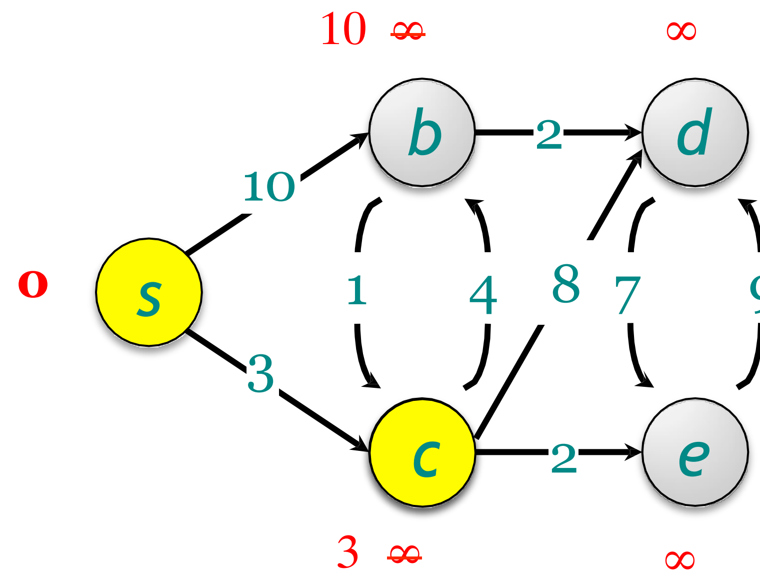

S={s}

c

Q={b, d, e}

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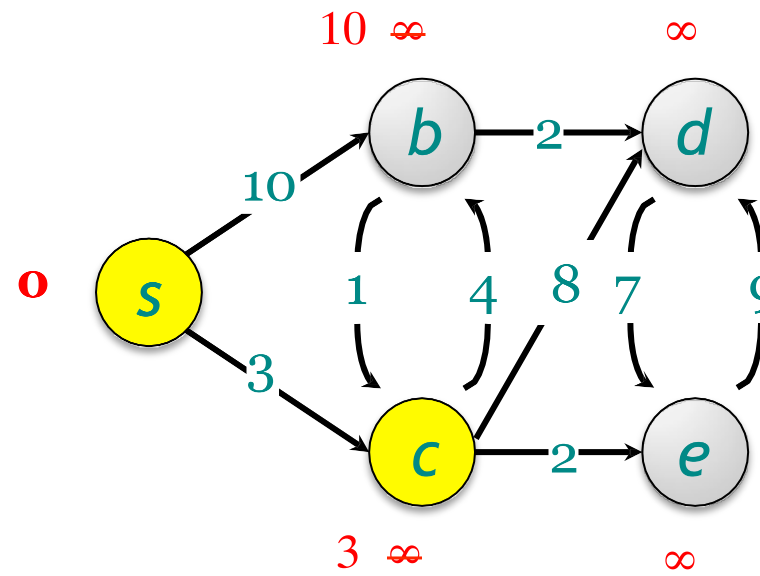

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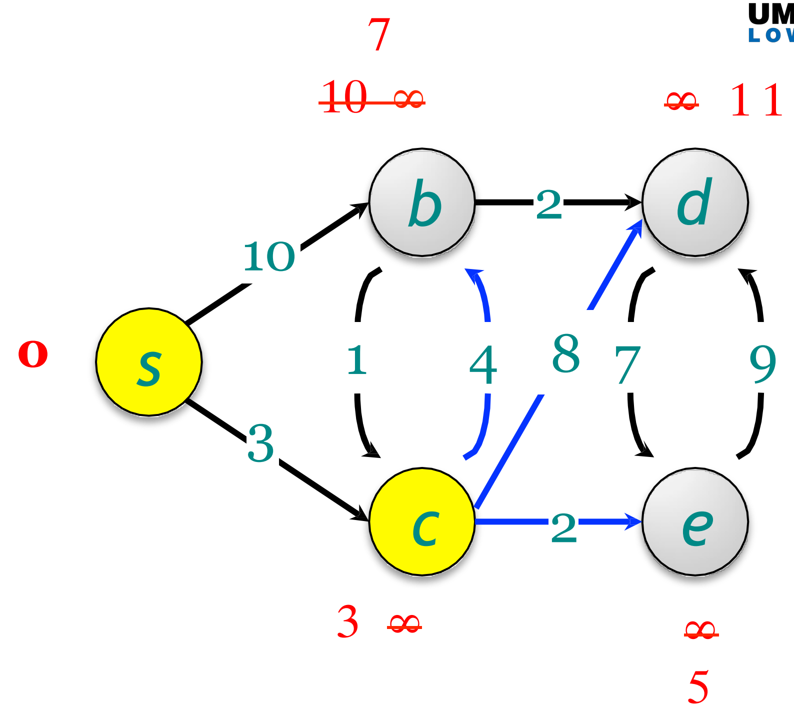

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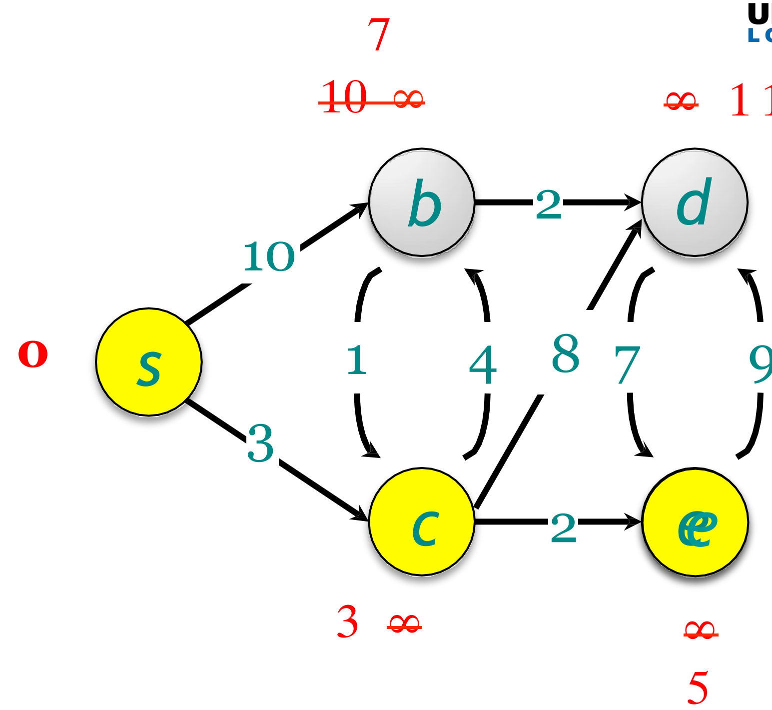

S={s, c}

e

Q={b, d}

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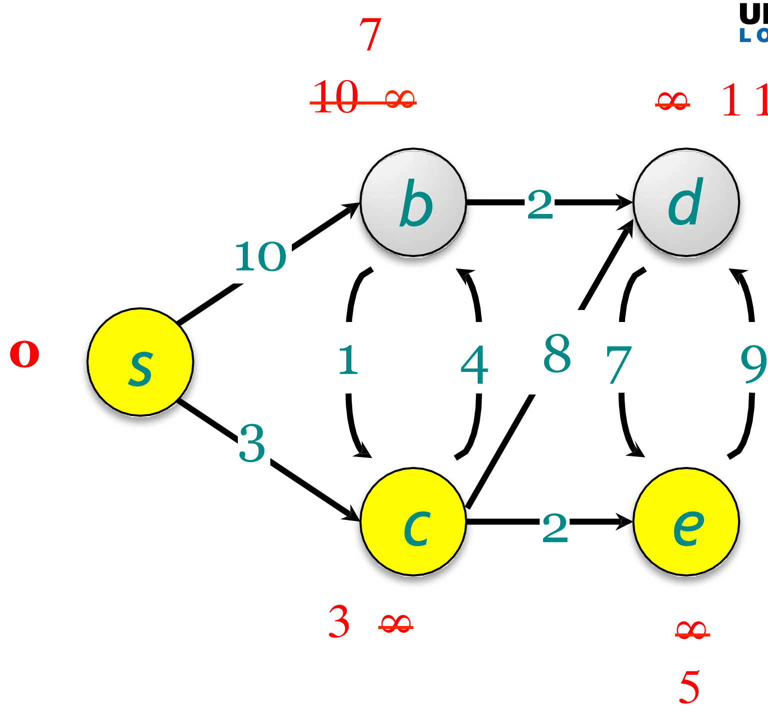

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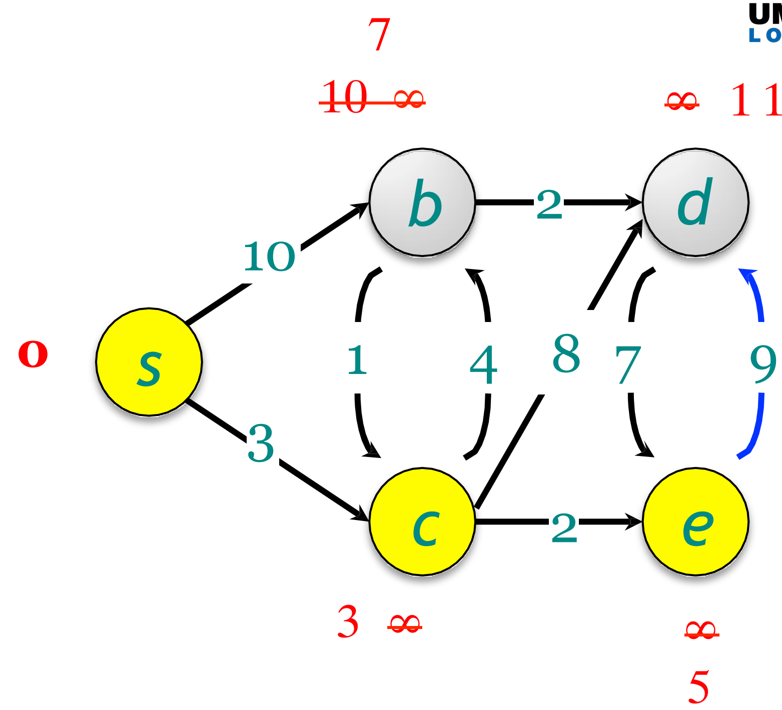

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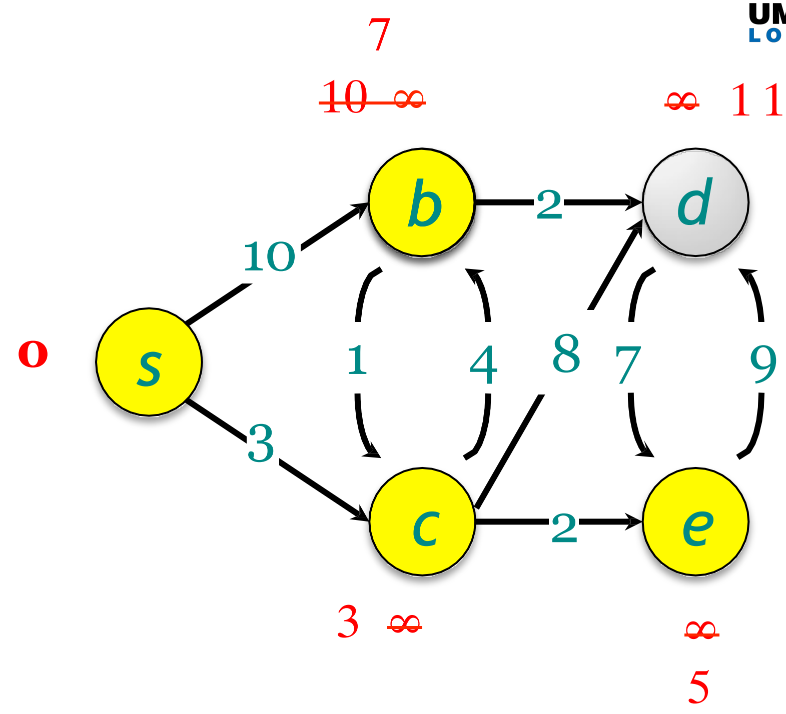

S={s, c, e}

b

Q={d}

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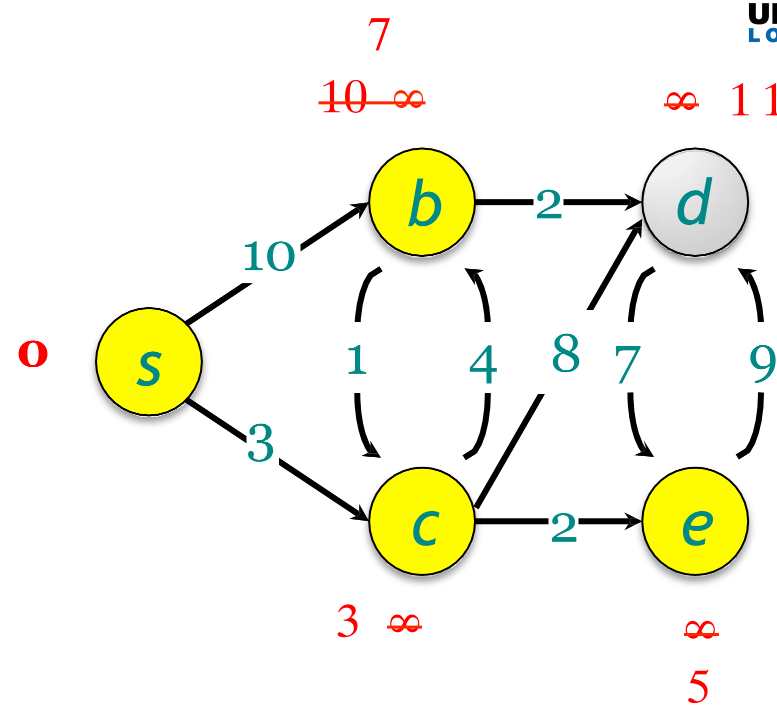

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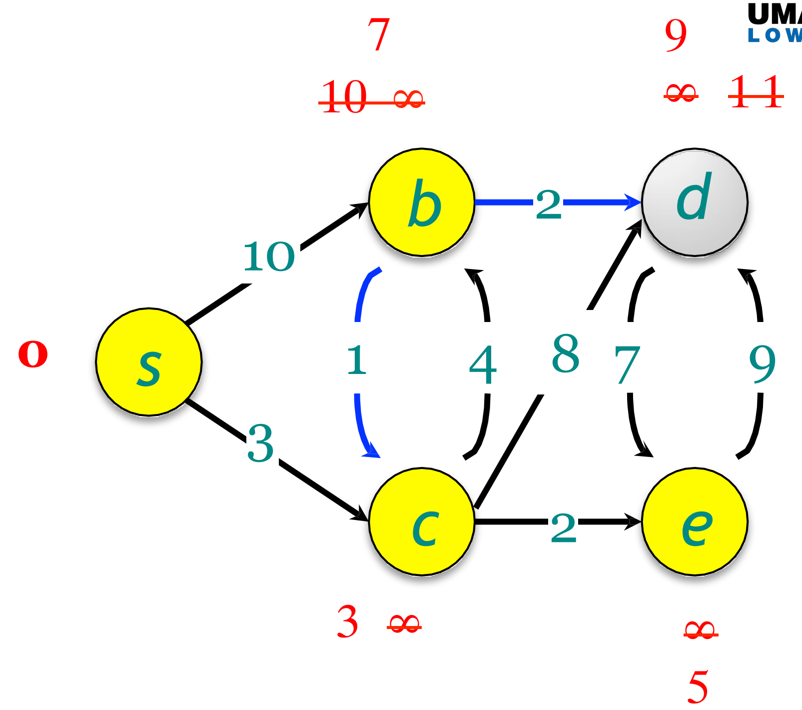

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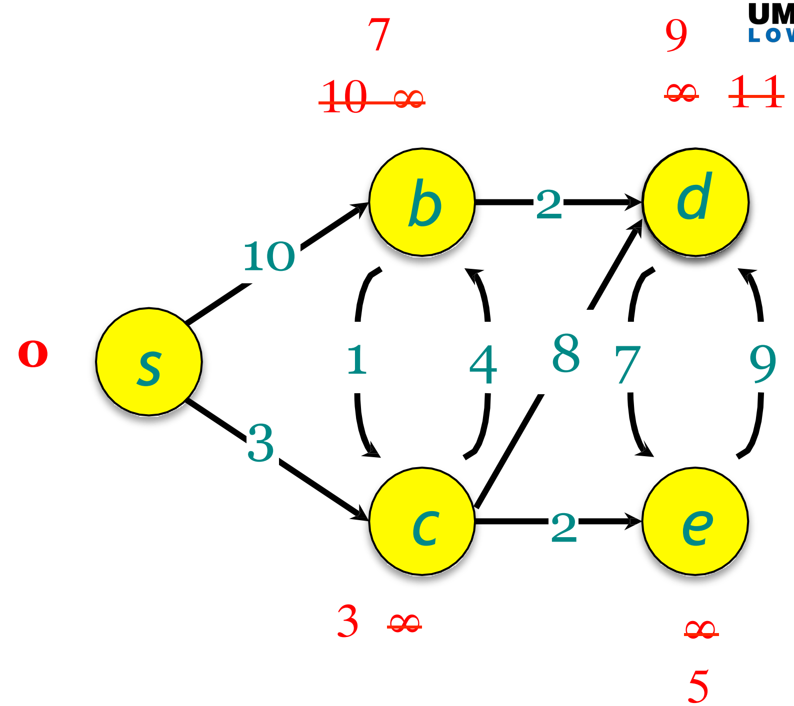

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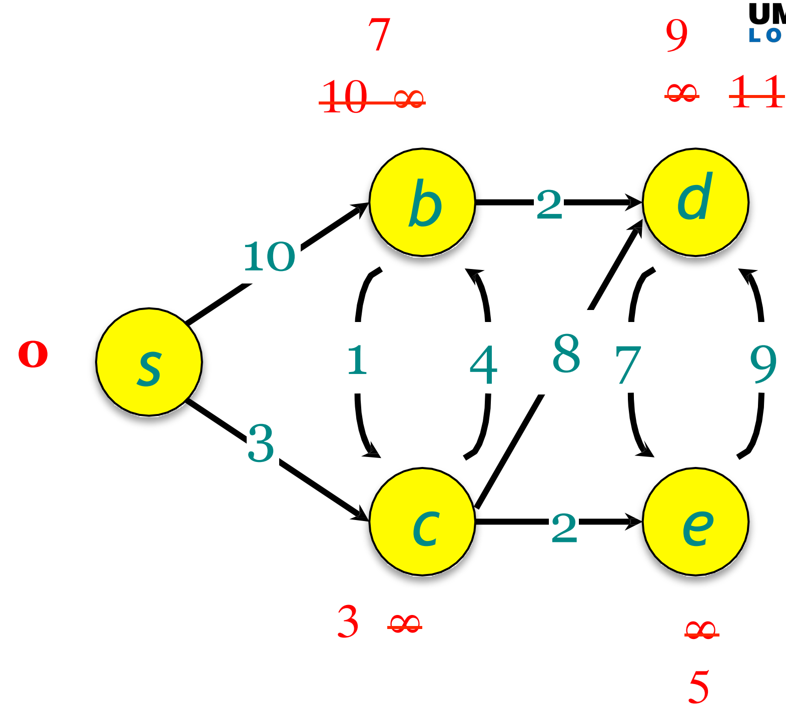

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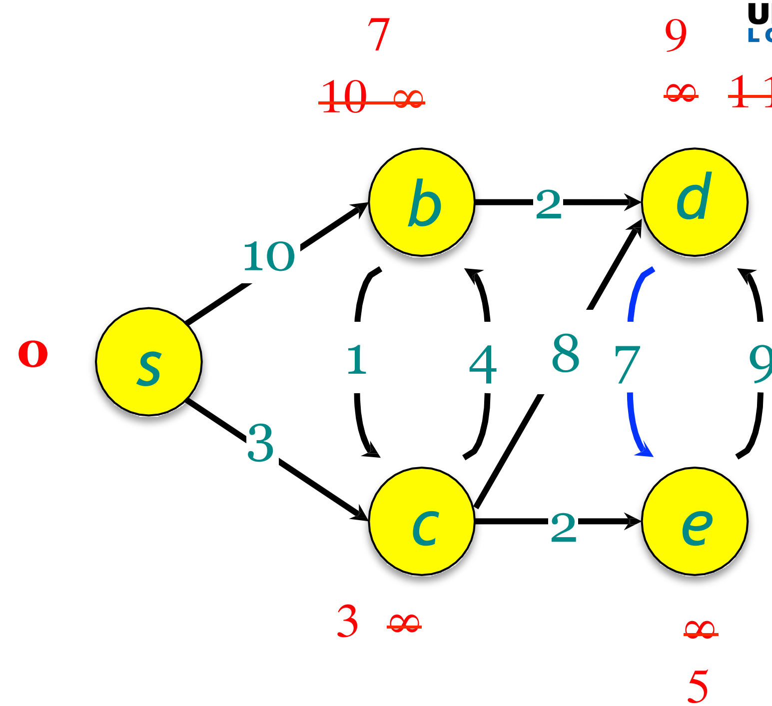

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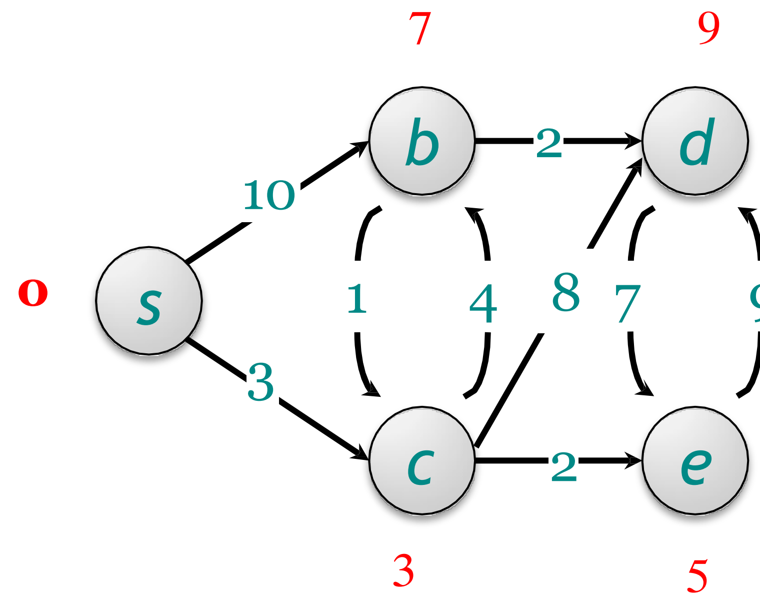

S={s, c, e, b, d}

Q={}

Dijkstra's algorithm- Cnt.

- Dijkstra's algorithm computes the shortest distances btw a start node and all other nodes in the graph (not only a target node)!

- Assumptions:
  - the graph is connected, and
  - the weights are nonnegative

Dijkstra's algorithm- Analysis

- $d(s) \leftarrow 0$
- **for** each $v \in \boldsymbol{N} - \{s\}$
  - **do** $d(v) \leftarrow \infty$
- $\boldsymbol{S} \leftarrow \varnothing$
- $\boldsymbol{Q} \leftarrow \boldsymbol{N}$
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degree (u) times

$|\boldsymbol{N}|$ times

Time = $\Theta(N \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{Relaxation}})$, Handshaking Lemma!

Dijkstra's algorithm- Analysis- Cnt.

$$\text{Time} = \Theta\left(N \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{Relaxation}}\right)$$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
Array	$O(N)$	$O(1)$	$O(N^2)$

Reading

- Ch.24 Single Source Shortest Paths [CLRS]

Network Basics 3

Graph ML

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University of Massachusetts, Lowell

Hadi Amiri
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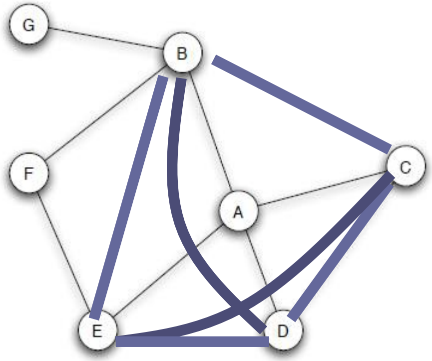
Lecture Topics

- Triadic closure and Bridges
- Neighborhood overlap
- The Strength of Weak Ties
- Structural Holes
- Node Centrality
- Edge Centrality
- Homophily
- Snapshot Algorithm
- Network Segregation

Triadic Closure

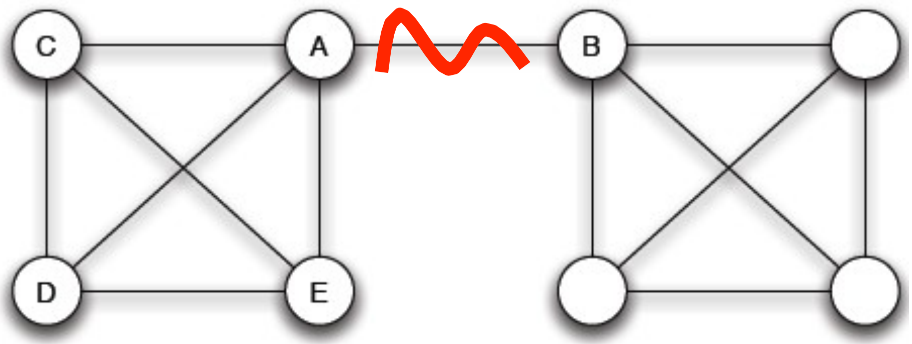
- If two **nodes** in a network have a **neighbor** in common, then there is an increased likelihood they will become **connected** themselves.
  - Reasons for Triadic Closure:
    - Opportunity, Trust, Incentives
- Clustering Coefficient
  - A measure to capture the prevalence of Triadic Closure
  - Defined for **nodes**



$$CF(A) = \frac{\textit{Number of connections btw A's friends}}{\textit{Possible Number of connections btw A's friends}} = 1/6$$

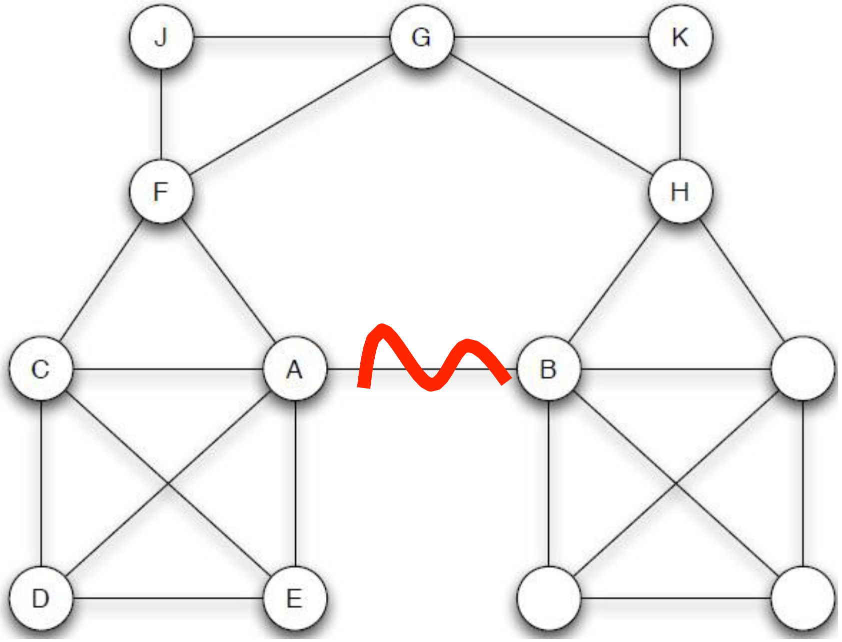
Bridge

- An edge is bridge if deleting it would put its two ends into two different connected components.
  - Bridges provide access to parts of the network that are unreachable by other means!



Local Bridge

- An edge such that its endpoints have no friends in common! → edge not in a triangle!
  - deleting a local bridge increases the distance btw its endpoints to a value strictly > **2**.



The Strength of Weak Ties

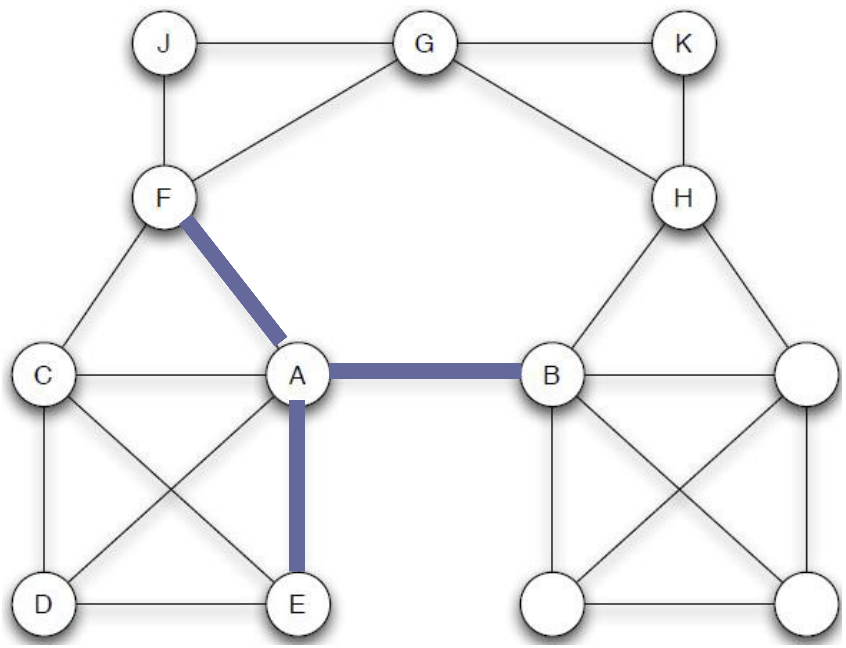
- Weak ties (acquaintances) connect us to new sources of information.
  - This dual role - as weak connections but also valuable links to hard-to-reach parts of the network - is the surprising strength of weak ties.

Neighborhood Overlap

- A measure to capture bridgeness of an edge!

$$\frac{\text{number of nodes who are neighbors of } \textit{both } A \text{ and } B}{\text{number of nodes who are neighbors of } \textit{at least one of } A \text{ or } B\text{'}}$$

Don't count A and B here!



Nodes	Neighborhood overlap
A-E	2/4
A-F	1/6
A-B	0/8 (**Overlap = 0** for local bridges)

Edges with very small neighborhood overlap can be considered as "almost" local bridges

Questions

1. Relation btw neighborhood overlap of an edge and its tie strength?

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   - Neighborhood overlap should grow as tie strength Grows.

Questions

2. How weak ties serve to link different communities that each contain large number of stronger ties?

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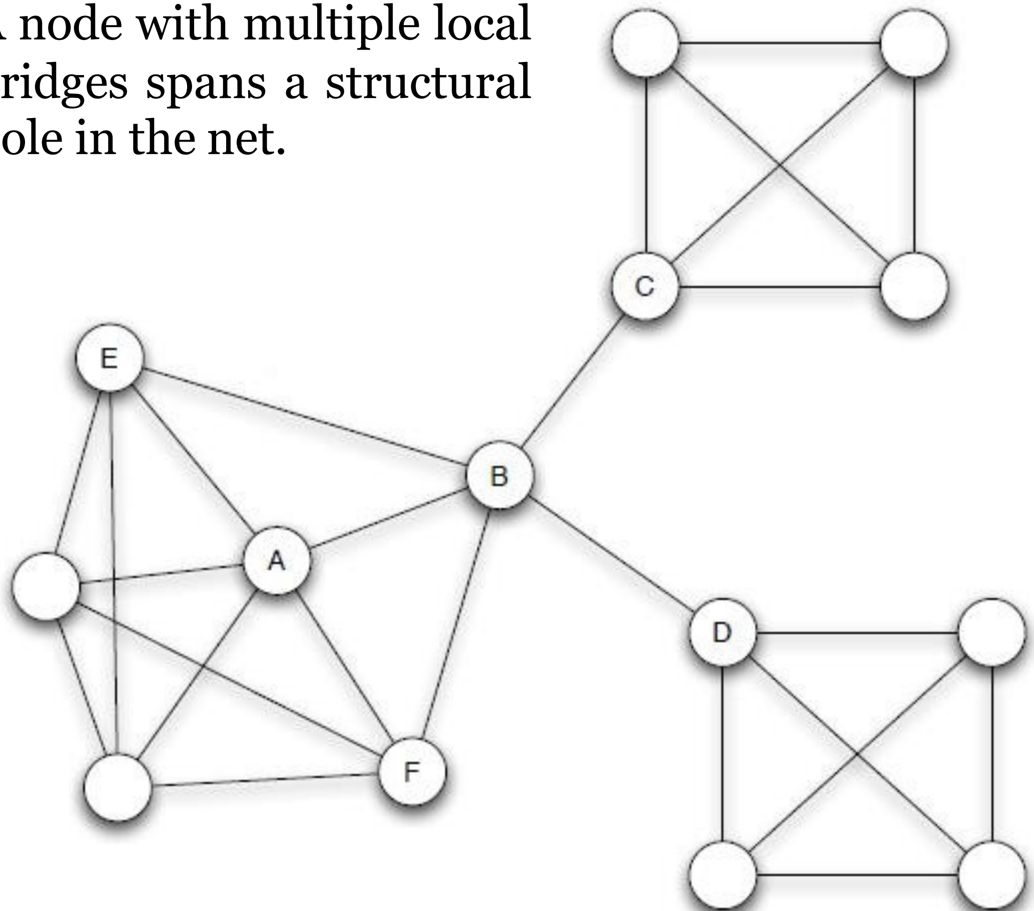
   - Delete edges from the network one at a time, start with the weakest ties first!
     - The giant component shrinks rapidly.

Structural Holes

Structural hole: the “empty space” in the net btw 2 sets of nodes that don’t interact closely!

A node with multiple local bridges spans a structural hole in the net.



B has early access to info!

B is a gatekeeper and controls the ways in which groups learn about info. She has power!

B may try to prevent triangles from forming around the local bridges she is part of!

How long these local bridges last before triadic closure produces short-cuts around them?

Node Centrality

- Degree centrality
  - A node is central if it has ties to many other nodes
- Closeness centrality
  - A node is central if it is “close” to other nodes
- Betweenness centrality
  - A node is central if other nodes have to go through it to get to each other

Edge Centrality

- Betweenness:
  - Let's assume 1 unit of "flow" will pass over all shortest path btw any pair of nodes A and B.
  - Betweenness of an edge is the total amount of flow it carries!
  - If there are *k* shortest path btw A and B, then 1/k units of flow will go along each shortest path!
- Girvan-Newman Algorithm:
  - Repeat until no edges are left:
    - Calculate betweenness of edges
    - Remove edges with highest betweenness

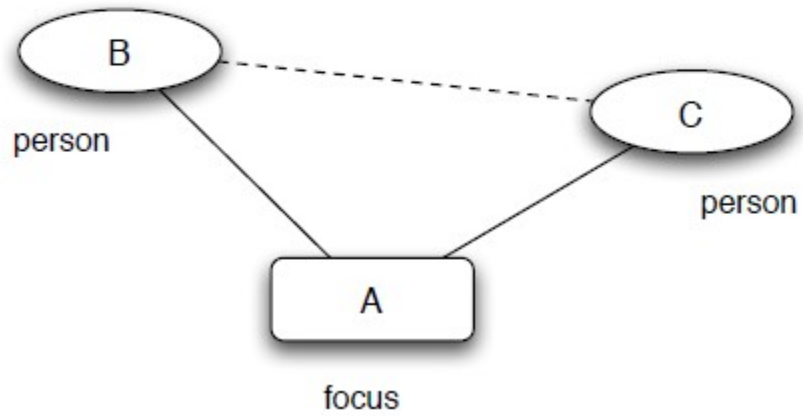
Homophily

- Links connect people with *similar* characteristics.
- Homophily has two mechanisms for link formation:
  - Selection:
    - Selecting friends with similar characteristics
      - Individual characteristics drive the formation of links
      - Immutable characteristics
  - Social Influence (socialization)
    - Modify behaviors to make them close to behaviors of friends
      - Existing links influence the individual characteristics of the nodes
      - Mutable characteristics

Homophily- Cnt.

- **Focal Closure**:
  B and C people, A focus

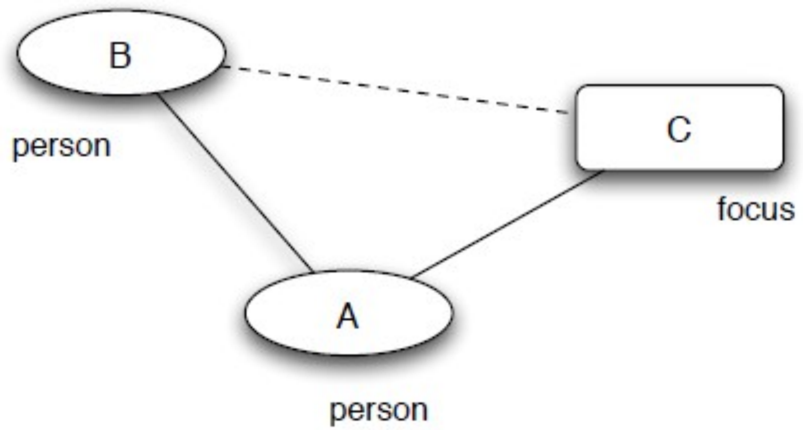
- **Selection**: B links to similar C (common focus)



(b) *Focal closure*

Homophily- Cnt.

- **Membership Closure**: A and B people, C focus
- **Social Influence**: B links to C influenced by A



Snapshot Algorithm

Tracking link formation in large scale datasets based on the above mechanisms

1) Take 2 snapshots of network at different times: **S(1)**, **S(2)**.

2) For each ***k***, find all pairs of nodes in **S(1)** that are not directly connected but have ***k*** common friends.

3) Compute ***T*(*k*)** as the fraction of these pairs connected in **S(2)**.

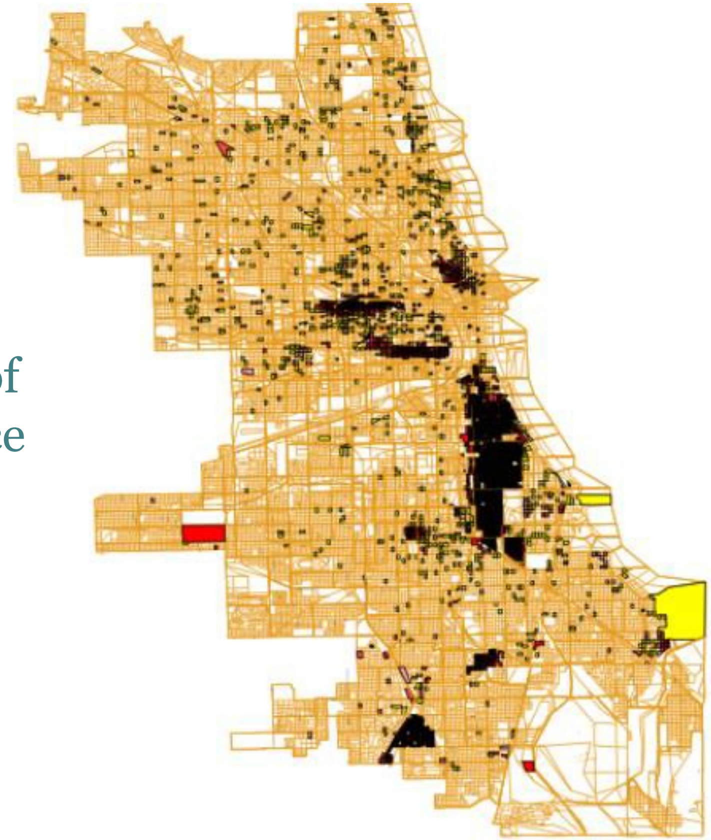
estimate for the probability that a link will form btw 2 people with *k* common friends.

4) Plot ***T*(*k*)** as a function of ***k***

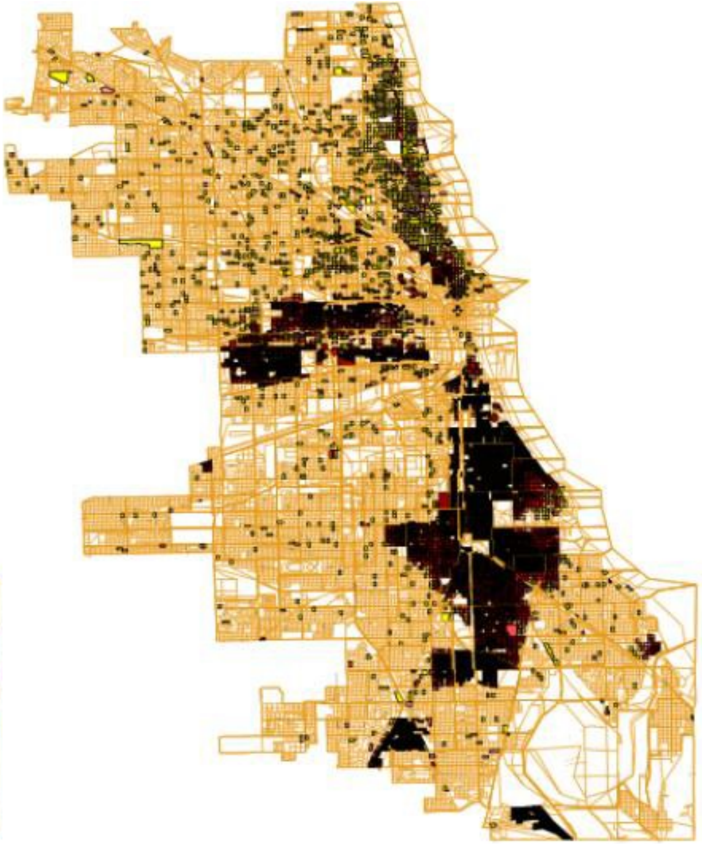
T(0) is the rate of link formation when it does not close a triangle

Spatial Model of Segregation

Schelling model
Local preferences of individuals can produce **unintended global patterns**.



(a) *Chicago, 1940*



(b) *Chicago, 1960*

Effects of homophily in the formation of ethnically and racially **homogeneous neighborhoods** in cities.

People live near others like them!!

Color the map wrt to a given race :

--Lighter: Lowest percentage of the race

--Darker: highest percentage of the race.

Questions?

(Optional) Reading

- Ch.02 Graphs [NCM]
- Ch.03 Strong and Weak Ties [NCM]